Introduction to Pre-Calculus Algebra

This module is a follow on from the Basic Algebra module.

Before you start this module you should be familiar with the following topics from the Basic Algebra module.

Introduction	Introduction to algebra and equations
Topic 2	Adding and subtracting like terms
Topic 3	Multiplying and dividing terms
Topic 4	Expanding brackets
Topic 5	Factorising

This module will review the content above and extend the concepts of Expanding, factorising and solving equations. The concept of inequalities will be covered.

A **polynomial** is made up of terms that are only added, subtracted or multiplied.

The word 'polynomial' suggests an expression that contains many terms, however, two commonly used terms are: binomial – 2 terms (for example: $2x+1$) and trinomial – 3 terms (for example: $2x^2-5x+1$).

It is common to see polynomials with just one variable, such as $x^4 + 2x^2 – 12$. With these polynomials it is common to write these in a standard form. In standard form, the term with the highest exponent is written first. Other terms are then written in exponent descending order.

The polynomial, $5x^3 – 7x^2 + x – 7$ is written in standard form. The highest exponent of any term in the polynomial is called the **degree** of the polynomial. For this polynomial, the degree is 3.

In this module some properties of real numbers are used.

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Addition and subtraction properties		
Addition with zero	Adding zero to a term will not change its value	$n + 0 = n$ $0 + n = n$
Subtraction with zero	Subtracting zero from a term will not change its value Subtracting a term from zero will give the inverse of the term	$n - 0 = n$ $0 - n = -n$ $0 - (-n) = n$ Take Care!
Commutative property of addition	The order in which terms are added will give the same result	$a + b = b + a$
Order of subtraction	Order cannot be changed for subtraction	$a - b \neq b - a$ But the rule below can apply $a - b = -(b - a)$
Associative property of addition	Numbers can be grouped in any order	$(a + b) + c = a + (b + c)$
Multiplication properties		
Multiplication with zero	When a term is multiplied by zero, the result is zero	$n \times 0 = 0$ $0 \times n = 0$
Multiplication with one	When a term is multiplies by 1 the result is the term	$n \times 1 = 1$ $1 \times n = 1$
Commutative property of multiplication	The order terms are multiplied will give the same result	$n \times m = m \times n$
Associative property of Multiplication	Numbers can be grouped in any order	$(a \times b) \times c = a \times (b \times c)$
Division properties		
Division by itself	When a term is divided by itself, the result is 1	$n \div n = 1$ $(n \neq 0)$
Division by One	When a term is divided by1, the result is itself	$n \div 1 = n$
One divided by a term	When 1 is divided by a term, the result is the reciprocal of the term	$1 \div n = \dfrac{1}{n}$ $1 \div \dfrac{m}{n} = \dfrac{n}{m} \quad \left(1 \div \dfrac{m}{n} = 1 \times \dfrac{n}{m} = \dfrac{n}{m} \right)$ $(m, n \neq 0)$
Division by zero	When a term is divided by Zero, the answer is undefined	$n \div 0 = undefined$
Zero divided by a term	When 0 is divided by a term the answer is 0	$0 \div n = 0$ $(n \neq 0)$

Topic 1: Multiplying (expanding) and dividing

Multiplying

Expanding is when the distributive law is used to remove brackets, taking a product and writing this as a sum. Factorising is the opposite of expanding, where a sum is written as a product.

Product   Sum

$$3(a+b) = 3a + 3b$$
$$\text{Expanding}$$

Sum   Product

$$3a + 3b = 3(a+b)$$
$$\text{Factorising}$$

To expand brackets, everything in the brackets must be multiplied by the term outside the brackets.

Multiplying by a constant
When an algebraic expression is multiplied by a constant, every term in the polynomial must be multiplied by the constant. If the constant is a negative, take care getting the sign of the answer. If more than one multiplication is being performed, finish the question by adding or subtracting like terms.

For example:

$2(f + 2g - i)$ $= 2 \times f \ + \ 2 \times 2g \ - \ 2 \times i$ $= 2f + 4g - 2i$	$-3(x^2 - 4x - 9)$ $= -3 \times x^2 \ - 3 \times -4x \ - 3 \times -9$ $= -3x^2 + 12x + 27$
$3(25 - 4x^2) - 5(2x^2 + 3x - 10)$ $= 75 - 12x^2 - 10x^2 - 15x + 50$ $= -12x^2 - 10x^2 - 15x + 50 + 75$ (Take Care! 75 is added) $= -22x^2 - 15x + 125$	$\frac{2}{3}(6a + 9b - 4)$ $= \frac{2}{3} \times 6a \ + \ \frac{2}{3} \times 9b \ - \ \frac{2}{3} \times 4$ $= 4a + 6b - \frac{8}{3}$

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Multiplying by a single term containing a variable(s)

When a polynomial is multiplied by a term containing a variable, multiplication of variables is required. When multiplying variables, skills in exponents may be required.

Examples:

$a(b+2c+8)$ $= a \times b \; + \; a \times 2c \; + \; a \times 8$ $= ab + 2ac + 8a$	$3x(x^2+x-3)$ $= 3x \times x^2 \; + \; 3x \times x \; + \; 3x \times {}^-3$ $= 3x^3 + 3x^2 - 9x$
$3ab(4a-b)$ $= 3ab \times 4a \; - \; 3ab \times b$ $= 12a^2b - 3ab^2$	$-3x^2(x^2-x+6)$ $= -3x^2 \times x^2 \; - \; 3x^2 \times {}^-x \; + \; {}^-3x^2 \times 6$ $= -3x^4 + 3x^3 - 18x^2$
$5m^3(am+bn-4)$ $= 5m^3 \times am \; + \; 5m^3 \times bn \; - \; 5m^3 \times 4$ $= 5am^4 + 5bm^3n - 20m^3$	$\dfrac{3}{x}\left(3x^4 - x^3 + \dfrac{x^2}{3} - 4x\right)$ $= \dfrac{3}{\cancel{x}} \times 3x^{\cancel{4}^3} \; - \; \dfrac{3}{\cancel{x}} \times x^{\cancel{3}^2} \; + \; \dfrac{\cancel{3}}{\cancel{x}} \times \dfrac{x^{\cancel{2}^1}}{\cancel{3}} \; - \; \dfrac{3}{\cancel{x}} \times 4\cancel{x}$ $= 9x^3 - 3x^2 + x - 12$

- Take care with signs
- Terms involving numbers (coefficients) and pronumerals are usually written with the number (coefficient) in front, followed by the variables in alphabetical order.

Multiplying one polynomial by another polynomial

When a polynomial is multiplied by another polynomial, it is important to use a method that ensures that every term from one polynomial is multiplied by every term in the other polynomial. After practice, you may be able to do these mentally.

To help in this process, one of two methods is used. Remember that when no operation is shown, multiplication is implied.

The **first method** involves multiplying the second polynomial by each individual term in the first polynomial. This is achieved by multiplying the second polynomial by each term of the first polynomial. These are multiplied out and like terms (if any) are collected. In the question $(a+3)(2b+1)$ the second term is multiplied by a and then by 3. This method works well in any situation.

$$(a+3)(2b+1)$$
$$= a(2b+1) + 3(2b+1)$$
$$= 2ab + a + 6b + 3$$

Note: There are no like terms so this cannot be simplified any further.

The **second method** involves multiplying every term in the first polynomial with every term in the second polynomial. If two binomials are used, the acronym FOIL is useful;

F – multiply the **F**irst terms
O – multiply the **O**uter terms
I – multiply the **I**nner terms
L – multiply the **L**ast terms

Multiply the **F**irst terms: *a x 2b =*
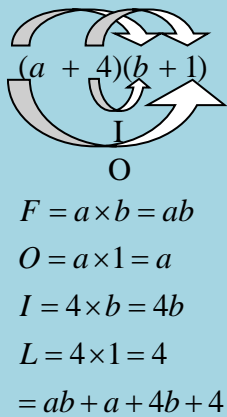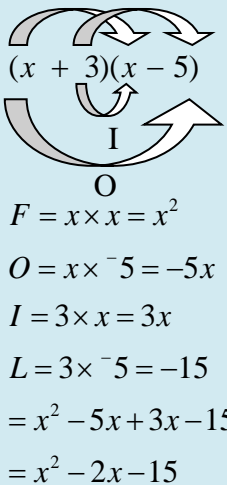
Multiply the **L**ast terms: *3 x 1 = 3*

$$(a+3)(2b+1)$$
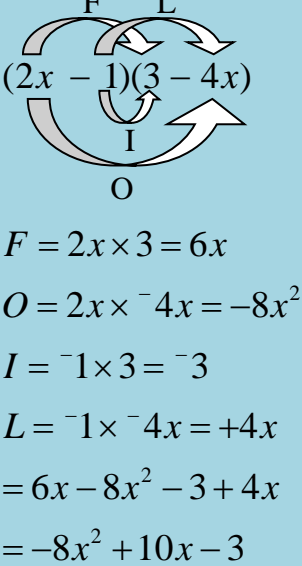
Multiply the **I**nside terms: *3 x 2b =*

Multiply the **O**utside terms: *a x 1*

$$(a+3)(2b+1) = 2ab + a + 6b + 3$$

Example: Expand

Method 1:	Method 2:
$(a+4)(b+1)$ Split the 1st factor into a and 4 $= a(b+1)+4(b+1)$ $= ab+a+4b+4$	$\overset{F \qquad L}{(a + 4)(b + 1)}$ $\underset{O}{\overset{I}{}}$ $F = a \times b = ab$ $O = a \times 1 = a$ $I = 4 \times b = 4b$ $L = 4 \times 1 = 4$ $= ab+a+4b+4$
$(x+3)(x-5)$ $= x(x-5)+3(x-5)$ $= x^2-5x+3x-15$ $= x^2-2x-15$	$\overset{F \qquad L}{(x + 3)(x - 5)}$ $\underset{O}{\overset{I}{}}$ $F = x \times x = x^2$ $O = x \times {}^-5 = -5x$ $I = 3 \times x = 3x$ $L = 3 \times {}^-5 = -15$ $= x^2-5x+3x-15$ $= x^2-2x-15$
$(2x-1)(3-4x)$ $= 2x(3-4x)-1(3-4x)$ $= 6x-8x^2-3+4x$ $= -8x^2+10x-3$	$\overset{F \qquad L}{(2x - 1)(3 - 4x)}$ $\underset{O}{\overset{I}{}}$ $F = 2x \times 3 = 6x$ $O = 2x \times {}^-4x = -8x^2$ $I = {}^-1 \times 3 = {}^-3$ $L = {}^-1 \times {}^-4x = +4x$ $= 6x-8x^2-3+4x$ $= -8x^2+10x-3$
$(2x-3)(x^2-2x+4)$ $= 2x(x^2-2x+4)-3(x^2-2x+4)$ $= 2x^3-4x^2+8x-3x^2+6x-12$ $= 2x^3-7x^2+14x-12$	The FOIL method only works (best) for two binomial factors.

$(x-2y+4z)(4x+y-2z)$ $= x(4x+y-2z)-2y(4x+y-2z)+4z(4x+y-2z)$ $= 4x^2+xy-2xz-8xy-2y^2+4yz+16xz+4yz-8z^2$ $= 4x^2-7xy+14xz-2y^2+8yz-8z^2$	
	$3(3x-5)(x^2-x-1)$ expand brackets first $= 3[3x(x^2-x-1)-5(x^2-x-1)]$ $= 3[3x^3-3x^2-3x-5x^2+5x+5]$ $= 3(3x^3-8x^2+2x+5)$ $= 9x^3-24x^2+6x+15$

The video below covers some examples similar to those above. Click on the image below to start downloading the video. Depending on your internet connection, this may take a minute or two.

Video 'Multiplying (expanding)'

Identities

Special binomial multiplications – Perfect Square

In this section, the special case of a binomial squared, written as $(a+b)^2$, is considered. Because this can be used for any binomial factor, it is given the special name of an identity.

Developing the identity:

$$(a+b)^2$$
$$= (a+b)(a+b)$$
$$= a(a+b)+b(a+b)$$
$$= a^2 + ab + ba + b^2$$
$$= a^2 + 2ab + b^2$$

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

When expanding a perfect square, there are two options:

(1) use the identity or (2) just expand using method 1.

For example:

Using the identity	Expanding using method 1
$(x+3)^2$ using the identity $a=x$, $b=3$ $= \underset{a^2}{x^2} + \underset{+2ab}{2 \times x \times 3} + \underset{+b^2}{3^2}$ $= x^2 + 6x + 9$	$(x+3)^2$ $= (x+3)(x+3)$ $= x(x+3)+3(x+3)$ $= x^2 + 3x + 3x + 9$ $= x^2 + 6x + 9$
$(3x+5)^2$ using the identity $a=3x$, $b=5$ $= \underset{a^2}{(3x)^2} + \underset{+2ab}{2 \times 3x \times 5} + \underset{+b^2}{5^2}$ $= 9x^2 + 30x + 25$	$(3x+5)^2$ $= 3x(3x+5)+5(3x+5)$ $= 9x^2 + 15x + 15x + 25$ $= 9x^2 + 30x + 25$
$(2x+5y)^2$ using the identity $a=2x$, $b=5y$ $= \underset{a^2}{(2x)^2} + \underset{+2ab}{2 \times 2x \times 5y} + \underset{+b^2}{(5y)^2}$ using brackets will ensure both coefficient and pronumeral are squared $= 4x^2 + 20x + 25y^2$	$(2x+5y)^2$ $= 2x(2x+5y)+5y(2x+5y)$ $= 4x^2 + 10xy + 10xy + 25y^2$ $= 4x^2 + 20x + 25y^2$

There is similar identity based on subtraction;

$$(a-b)^2$$
$$= (a-b)(a-b)$$
$$= a(a-b) - b(a-b)$$
$$= a^2 - ab - ba + b^2$$
$$= a^2 - 2ab + b^2$$

$$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

$(x-7)^2$ using the identity $a = x,\ b = 7$ $= \underset{a^2}{x^2} - \underset{-2ab}{2 \times x \times 7} + \underset{+b^2}{7^2}$ $= x^2 - 14x + 49$	$(x-7)^2$ $= (x-7)(x-7)$ $= x(x-7) - 7(x-7)$ $= x^2 - 7x - 7x + 49$ $= x^2 - 14x + 49$
$(4x - y)^2$ using the identity $a = 4x,\ b = y$ $= \underset{a^2}{(4x)^2} - \underset{+2ab}{2 \times 4x \times y} + \underset{+b^2}{y^2}$ $= 16x^2 - 8xy + y^2$	$(4x - y)^2$ $= 4x(4x - y) - y(4x - y)$ $= 16x^2 - 4xy - 4xy + y^2$ $= 16x^2 - 8xy + y^2$

Although using identities can be useful, the most important skill is knowing how to expand logically as this will give the skills to cope with any given situation.

Take care:

<div style="background-color: yellow">

A commonly made mistake is when expanding a perfect square is:
$$(x+2)^2 \neq x^2 + 4$$
This is incorrect because the middle term is omitted.

</div>

An example that combines the skills covered so far are:

Expand $3x(2x-3)(x+2)^2$

With complex expansions like this, start with the perfect square, then multiply by the binomial factor and then the factor _3x_.

$(x+2)^2$

$= (x+2)(x+2)$

$= x(x+2) + 2(x+2)$

$= x^2 + 4x + 4$

Now the expression is: $3x(2x-3)(x^2+4x+4)$

$(2x-3)(x^2+4x+4)$

$= 2x(x^2+4x+4) - 3(x^2+4x+4)$

$= 2x^3 + 8x^2 + 8x - 3x^2 - 12x - 12$

$= 2x^3 + 5x^2 - 4x - 12$

Now the expression is: $3x(2x^3 + 5x^2 - 4x - 12)$

$3x(2x^3 + 5x^2 - 4x - 12)$

$= 3x \times 2x^3 + 3x \times 5x^2 - 3x \times 4x - 3x \times 12$

$= 6x^4 + 15x^3 - 12x^2 - 36x$

There is one more identity worth knowing about. This identity is about the product of two binomial factors in the form of $\underset{sum}{(a+b)} \underset{difference}{(a-b)}$

Expanding these two binomial factors gives:

$$(a+b)(a-b)$$
$$= a(a-b) + b(a-b)$$
$$= a^2 - ab + ab - b^2$$
$$= a^2 - b^2$$

For example:

$(x-3)(x+3)$ $(a-b)(a+b)$ $= x^2 - 3^2$ a^2-b^2 $= x^2 - 9$	$(x-3)(x+3)$ $= x(x+3) - 3(x+3)$ $= x^2 + 3x - 3x - 9$ $= x^2 - 9$
$(5+2x)(5-2x)$ $(a+b)(a-b)$ $= 5^2 - (2x)^2$ a^2-b^2 $= 25 - 4x^2$	$(5+2x)(5-2x)$ $= 5(5-2x) + 2x(5-2x)$ $= 25 - 10x + 10x - 4x^2$ $= 25 - 4x^2$

More examples that combine the skills cover so far are:

Expand $x^2(5-3x)(5+3x)$

$$(5-3x)(5+3x)$$
$$= 25 - 9x^2$$

Now the expression is:

$$x^2(25 - 9x^2)$$
$$= 25x^2 - 9x^4$$

Expand $(3x-1)^3$

this is the same as $(3x-1)(3x-1)^2$

$$(3x-1)^2$$
$$= 9x^2 - 6x + 1$$

Now the expression is:

$$(3x-1)(9x^2 - 6x + 1)$$
$$= 3x(9x^2 - 6x + 1) - 1(9x^2 - 6x + 1)$$
$$= 27x^3 - 18x^2 + 3x - 9x^2 + 6x - 1$$
$$= 27x^3 - 27x^2 + 9x - 1$$

Expand $\dfrac{2}{3}\left(2x-\dfrac{1}{2}\right)\left(2x+\dfrac{1}{2}\right)(3x-4)^2$

Start with the more complex. There is a perfect square $(a-b)^2$ **and a**
$\underset{\text{sum}}{(a+b)}\,\underset{\text{difference}}{(a-b)}$ **structure. These are equally complex so order for these two parts is unimportant.**

$$(3x-4)^2$$
$$= 9x^2 - 24x + 16$$

Now the expression is $\dfrac{2}{3}\left(2x-\dfrac{1}{2}\right)\left(2x+\dfrac{1}{2}\right)\left(9x^2-24x+16\right)$

$$\left(2x-\dfrac{1}{2}\right)\left(2x+\dfrac{1}{2}\right)$$
$$= 4x^2 - \dfrac{1}{4}$$

Now the expression is $\dfrac{2}{3}\left(4x^2-\dfrac{1}{4}\right)\left(9x^2-24x+16\right)$

$$= \dfrac{2}{3}\left[4x^2\left(9x^2-24x+16\right)-\dfrac{1}{4}\left(9x^2-24x+16\right)\right]$$
$$= \dfrac{2}{3}\left[36x^4-96x^3+64x^2-\dfrac{9}{4}x^2+\dfrac{24}{4}x-\dfrac{16}{4}\right]$$
$$= \dfrac{2}{3}\left(36x^4-96x^3+\dfrac{247}{4}x^2+6x-4\right)$$
$$= 24x^4-64x^3+\dfrac{247}{6}x^2+4x-\dfrac{8}{3}$$

The video below covers some examples similar to those above. Click on the image below to start downloading the video. Depending on your internet connection, this may take a minute or two.

 Video 'Multiplying (Using identities)'

Polynomial division

In this section, a polynomial (dividend) will be divided by another polynomial (divisor) using a method similar to long division. The resulting answer is not necessarily a polynomial. It may be worth reviewing long division with numbers to help follow the example below.

Example: Dividing by a single term (monomial)

$$(5x^3 - 2x^2 + 4x) \div x$$

This is best written as

$$\frac{5x^3 - 2x^2 + 4x}{x}$$

$$= \frac{5x^3}{x} - \frac{2x^2}{x} + \frac{4x}{x}$$

$$= 5x^2 - 2x + 4$$

Example: Dividing by a single term (monomial)

$$\frac{(2x-3) - 2(3-2x)}{2x}$$

$$= \frac{2x - 3 - 6 + 4x}{2x}$$

$$= \frac{6x - 9}{2x}$$

$$= \frac{6x}{2x} - \frac{9}{2x}$$

$$= 3 - \frac{9}{2x}$$

Example: Dividing by a single term (monomial)

$$\frac{3xy^2 + 5x^2y - 15x^2y^2}{3x^2y^2}$$

$$= \frac{3xy^2}{3x^2y^2} + \frac{5x^2y}{3x^2y^2} - \frac{15x^2y^2}{3x^2y^2}$$

$$= \frac{1}{x} + \frac{5}{3y} - 5$$

When dividing by a binomial factor, the process is similar to performing long division with numbers.

Example:

$$6)\overline{4926} \to 6)\overline{4\underline{9}26} \;\; \to \;\; 6)\overline{4926}^{\,82} \to 6)\overline{4926}^{\,821}$$

$$\begin{array}{c} \underline{48}\downarrow \\ 1 \end{array} \qquad \begin{array}{c} \underline{48}\downarrow \\ 12 \\ \underline{12}\downarrow \\ 0 \end{array} \qquad \begin{array}{c} \underline{48}\downarrow \\ 12 \\ \underline{12}\downarrow \\ 0\;6 \\ \underline{6} \\ 0 \end{array}$$

Example: $(4x^2 + 4x + 1) \div (2x - 3)$

$2x - 3)\overline{4x^2 + 4x + 1}$ • think: what is $4x^2 \div 2x \; (= 2x)$

This gives;

$$2x - 3)\overline{4x^2 + 4x + 1}^{\;\;2x}$$ • Now subtract the product of $2x$ & $2x - 3$ ($= 4x^2 - 6x$)

$$\begin{array}{r} -\quad \underline{4x^2 - 6x} \\ 10x \end{array}$$ • Subtract $4x - (-6x) = 10x$

Now, consider the next term in the dividend;

$$2x - 3)\overline{4x^2 + 4x + 1}^{\;\;2x+5}$$ • bring down the next term

$$\begin{array}{r} -\quad \underline{4x^2 - 6x}\;\;\downarrow \\ 10x + 1 \\ -\quad \underline{10x - 15} \\ 16 \end{array}$$

• think: what is $10x \div 2x$? $(= 5)$
• multiply 5 and $2x - 3$ ($= 10x - 15$)
• subtract $1 - (-15) = 16$

The answer is expressed as $(4x^2 + 4x + 1) \div (2x - 3) = 2x + 5 + \dfrac{16}{2x - 3}$

Example: $(x^3 + 2x^2 - 5x - 6) \div (x + 1)$

$x + 1 \overline{)x^3 + 2x^2 - 5x - 6}$ • think: what is $x^3 \div x$ $(= x^2)$

This gives;

$$\begin{array}{r} x^2 \\ x+1 \overline{)x^3 + 2x^2 - 5x - 6} \\ - \ \underline{x^3 + x^2} \\ x^2 \end{array}$$

• Now subtract the product of x^2 & $x + 1$ $(= x^3 + x^2)$

• Subtract $2x^2 - x^2 = x^2$

Now, consider the next term in the dividend;

$$\begin{array}{r} x^2 + x \\ x+1 \overline{)x^3 + 2x^2 - 5x - 6} \\ - \ \underline{x^3 + x^2} \quad \downarrow \\ x^2 - 5x \\ - \ \underline{x^2 + x} \\ -6x \end{array}$$

• bring down the next term

• think: what is $x^2 \div x$? $(= x)$

• multiply x and $x + 1$ $(= x^2 + x)$

• subtract $-5x - x = -6x$

Now, consider the next term in the dividend;

$$\begin{array}{r} x^2 + x \ -6 \\ x+1 \overline{)x^3 + 2x^2 - 5x - 6} \\ - \ \underline{x^3 + x^2} \quad \downarrow \ \downarrow \\ x^2 - 5x \quad \downarrow \\ - \ \underline{x^2 + x} \quad \downarrow \\ -6x - 6 \\ - \ \underline{-6x - 6} \\ 0 \end{array}$$

• bring down the next term

• think: what is $-6x \div x$? $(= -6)$

• multiply -6 and $x + 1$ $(-6x - 6)$

• subtract

There is no remainder, so $(x^3 + 2x^2 - 5x - 6) \div (x + 1) = x^2 + x - 6$

With numbers; we can say that 8 is a factor of 96 because $96 \div 8 = 12$ with no remainder. In the example above; $(x^3 + 2x^2 - 5x - 6) \div (x + 1)$ gave the result $x^2 + x - 6$ with no remainder, therefore it is possible to say that $(x + 1)$ is a factor of $(x^3 + 2x^2 - 5x - 6)$.

The video below covers some examples similar to those above. Click on the image below to start downloading the video. Depending on your internet connection, this may take a minute or two.

Video 'Dividing (Polynomial division)'

Activity

1. Expand the following, collect like terms if possible.

(a) $3(9-2x)$

(b) $\dfrac{1}{2}(4x^2-x-9)$

(c) $4x-3(x-4)$

(d) $2(x^2+x+4)+3(x+6)-(x^2-2x+9)$

(e) $75-4(2x^2+x-6)$

(f) $4(x+4y)-5(x-2y+1)+2(x-3)$

2. Expand the following, collect like terms if possible.

(a) $a(3a-4b+1)$

(b) $4ac(b+2c-7)$

(c) $p^2\left(2p+4+\dfrac{5}{p}\right)$

(d) $x(x^2+4x-1)-3(x^2+1)$

(e) $-4x(9-x)-\dfrac{1}{2}(4-x)$

(f) $\dfrac{4}{3x}(6x^3+5x^2-x+4)$

3. Expand the following, collect like terms if possible.

(a) $(2a-1)(4b-3)$

(b) $(3x-2)(y+4)$

(c) $(x+3)(x-2)$

(d) $(x-4)(2x+1)$

(e) $(x-4)^2$

(f) $(2x-3)(7-x)$

(g) $(x-2)(x+2)$

(h) $(2x+1)(x-3)(x+2)$

(i) $(x-2)^3$

(j) $4(2x-1)(x^2+7x+5)$

4. Perform the following divisions.

(a) $(x^2-4x+11)\div(x+4)$

(b) $(x^3+4x^2+x-6)\div(x-1)$

(c) $(2x^3-5x^2-14x+8)\div(2x-1)$

The answers to the activity questions are found in the pdf named:

'Answers to Activity Questions (PCA)'

Topic 2: Factorisation

Factorising is the opposite of expanding. Factorising (sometimes called factoring) is based on the word Factors, so factorising is about producing factors. Just like the factor pairs of 36 will multiply to give 36, the factor of a polynomial must multiply to give the polynomial.

Sum	Product

$$3a + 3b = 3(a + b)$$
Factorising

Factorising is about starting with a sum (or difference) of terms and finishing with a product of factors. In the question above, the start is a sum(+) of two terms($3a, 3b$) and finishing with a product (x) of two factors [3 and $(a+b)$].

Factorising by common factor

Factorise $2a + 6$

The first step is to look at the two terms and identify the highest common factor (HCF). In this case the HCF is 2. Write each term with 2 as a factor. With experience, some of the steps may be omitted.

$$2a + 6$$
$$= 2 \times a + 2 \times 3$$
$$= 2(a + 3)$$

Examples

Factorise	H.C.F.	Solution
$3f - 21$	3	$3f - 21$ $= 3 \times f - 3 \times 7$ $= 3(f - 7)$
$2ab - a$	a	$2ab - a$ $= a \times 2b - a \times 1$ $= a(2b - 1)$
$3xy - 6xz$	$3x$	$3xy - 6xz$ $= 3x \times y - 3x \times 2z$ $= 3x \times (y - 2z)$ $= 3x(y - 2z)$

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$ab^2c + 3ab$	ab	$ab^2c + 3ab$ $= ab \times bc + ab \times 3$ $= ab(bc + 3)$
$^-25x^2 - 10x$ **Hint: take out a negative common factor if the first term is negative**	^-5x	$^-25x^2 - 10x$ $= {}^-5x \times 5x + {}^-5x \times 2$ $= {}^-5x(5x + 2)$

Remember, you can check your answer by expanding the answer you obtained to get the question back. For example:

$$ab(bc + 3)$$
$$= ab \times bc + ab \times 3$$
$$= ab^2c + 3ab$$
$$= \text{ the original expression}$$

Factorising by common factor (grouping factor)

Consider the question:

Factorise $x(a + b) + 4(a + b)$

You notice that this expression is basically two terms. Looking at both terms you will notice that $(a+b)$ is common to both terms. This common factor is called a grouping factor. It is a factor just like the questions above.

$$x(a + b) + 4(a + b)$$
$$= (a + b)(x + 4)$$

Factorise $p(r - s) - 4(r - s)$

$$p(r - s) - 4(r - s)$$
$$= (r - s)(p - 4)$$

Factorise $2e(a - b) - 5(b - a)$

Notice in this example that the grouping factors are not the same, but they are similar. Factorising by grouping factor cannot happen until they are both the same.
The binomial (b - a) can be rewritten as $-(a$ - $b)$. Check by expanding if you are unsure.

$$2e(a - b) - 5(b - a)$$
$$= 2e(a - b) - 5 \times -(a - b)$$
$$= 2e(a - b) + 5(a - b)$$
$$= (a - b)(2e + 5)$$

Factorise $y(a-b)+a-b$

This question becomes clearer if some extra detail is added, *a-b* can also be expressed as *1(a-b)*. Then the expression becomes:

$$y(a-b)+a-b$$
$$= y(a-b)+1(a-b)$$
$$= (a-b)(y+1)$$

Factorise $ab(a-b)+b-a$

$$ab(a-b)+b-a$$
$$= ab(a-b)+(b-a)$$
$$= ab(a-b)-1(a-b)$$
$$= (a-b)(ab-1)$$

Often the questions require some factorising before a grouping factor becomes clear.
For example: Factorise $ab+4b-4a-16$

By looking at the first two terms you notice a common factor of *b* and another factor (*a+4*). When looking at the last two terms you notice a common factor of *-4* and another factor (*a+4*).

$$ab+4b-4a-16$$
$$= b(a+4)-4(a+4)$$
$$= (a+4)(b-4)$$

Factorise $a^2-3ab-3a+9b$

$$a^2-3ab-3a+9b$$
$$= a(a-3b)-3(a-3b)$$
$$= (a-3b)(a-3)$$

Factorise $15pr-3rq+5ps-qs$

$$15pr-3rq+5ps-qs$$
$$= 3r(5p-q)+s(5p-q)$$
$$= (5p-q)(3r+s)$$

Factorising by common factor (exponents)

Sometimes a good knowledge of exponents is required to determine the common factor. In the example below the common factor is a power. Remember you can always expand to see if you are correct.

☞ Factorise $m^7 + m^4$

$$m^7 + m^4$$
$$= m^4(m^3 + 1)$$

☞ Factorise $m^a + m^{a+2}$

$$m^a + m^{a+2}$$
$$= m^a(1 + m^2)$$

☞ Factorise $e^{2x} + e^x$

$$e^{2x} + e^x$$
$$= e^x(e^2 + 1)$$

☞ Factorise $e^x + e^{-x}$

$$e^x + e^{-x}$$
$$= e^{-x}(e^{2x} + 1)$$
or
$$e^x + e^{-x}$$
$$= e^x(1 + e^{-2x})$$

Factorising trinomials

In this section we will practise factorising quadratic trinomials. Trinomials are polynomials that contain three terms. A quadratic trinomial is a trinomial in the form: $ax^2 + bx + c$ where *a, b and c (a≠0)* are coefficients, for example: $3x^2 - 4x + 1$.

This is an important skill to have for simplifying algebraic expressions and graphing functions.

Two methods are shown. The first method is logical where thinking about possible solutions is required. The second method is a sequence of steps, so following the method to achieve an answer just involves remembering the method.

Example: Factorise the trinomial $x^2 + 7x + 10$

Mental Strategy	Sequence of Steps Strategy
You know that a trinomial may produce 2 monomial factors. **Using ideas from the FOIL method of expanding, the <u>first term</u> of the trinomial, x^2, came from $x \times x$;** **the <u>last term</u> 10 could come from the factor pair 2 and 5, or 1 and 10.** **Construct a t diagram to help.** **On the left, list the factors of the first term x^2.** **On the right list the factors of the last term 10. (In this case: 2 & 5 and 1 & 10)** **The arrows represent the two middle terms (the OI from FOIL). Under each pair write sum of the two middle terms indicated by the arrow.** *(x+2)* *(x+5)* $2x+5x = 7x$ $1x+10 = 11x$ The middle term in the question is *7x*, so the result is: $x^2 + 7x + 10 = (x+2)(x+5)$	The general form of a trinomial is $ax^2 + bx + c$. So for the trinomial being factorised $x^2 + 7x + 10$; $ac = 1 \times 10 = 10$ and $b = 7$. Think of two numbers that multiply to 10 and add to 7? 2 and 5. If there are no such numbers, then it means it cannot be factorised. This gives us the breakdown of the middle term $7x$ into two separate terms $2x$ and $5x$. Now the trinomial becomes: $$x^2 + 7x + 10$$ $$= x^2 + 2x + 5x + 10$$ Using our knowledge of grouping factors: $$x^2 + 7x + 10$$ $$= x^2 + 2x + 5x + 10$$ $$= x(x+2) + 5(x+2)$$ $$= (x+2)(x+5)$$ If no grouping factor is found – a mistake has been made.

After some practice, you may use the Mental Strategy for easier questions and the Sequence Strategy for more difficult questions.
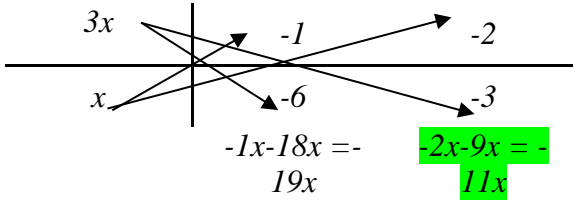
Example: Factorise the trinomial $x^2 + 2x - 15$

Mental Strategy	Sequence of Steps Strategy
The first term of the trinomial, x^2, came from $x \times x$. To obtain (-15) from two numbers, one will be positive, the other negative; so the factor pairs of (-15) are 1 and -15, -1 and 15, 3 and -5 or -3 and 5. Note factors that differ only by sign must be included. $1x-15x$ $=-14x$ $x+15x$ $= 14x$ $3x-5x$ $=-2x$ $3x+5x$ $= 2x$ This indicates the correct option is: $x^2 + 2x - 15 = (x-3)(x+5)$	For the trinomial being factorised $x^2 + 2x - 15$; $ac = 1 \times {}^-15 = {}^-15$ and $b = 2$. Two numbers that multiply to -15 and add to 2 are -3 and +5 (-3 x 5 = -15, -3 + 5 = 2) This gives us the breakdown of the middle term $2x$ into two separate terms $-3x$ and $5x$. Now the trinomial becomes: $$x^2 + 2x - 15$$ $$= x^2 - 3x + 5x - 15$$ $$= x(x-3) + 5(x-3)$$ $$= (x-3)(x+5)$$

Example: Factorise the trinomial $2x^2 + 3x - 2$

Mental Strategy	Sequence of Steps Strategy
The first term of the trinomial, $2x^2$, came from $2x \times x$. To obtain (-2) from two numbers, one will be positive, the other negative; so the factor pairs of (-2) are 1 and -2, -1 and 2, -2 and 1 or 2 and -1. $1x+4x$ $=3x$ $1x-4x$ $= -3x$ $2x+2x$ $= 0$ $2x-2x$ $=0$ This indicates the correct option is: $2x^2 + 3x - 2 = (2x-1)(x+2)$	For the trinomial being factorised $2x^2 + 3x - 2$; $ac = 2 \times {}^-2 = {}^-4$ and $b = 3$. Two numbers that multiply to -4 and add to 3 are -1 and +4 (-1 x 4 = -4, -1 + 4 = 3) This gives us the breakdown of the middle term $3x$ into two separate terms $-x$ and $4x$. Now the trinomial becomes: $$2x^2 + 3x - 2$$ $$= 2x^2 - x + 4x - 2$$ $$= x(2x-1) + 2(2x-1)$$ $$= (2x-1)(x+2)$$

Example: Factorise the trinomial $3x^2 - 11x + 6$
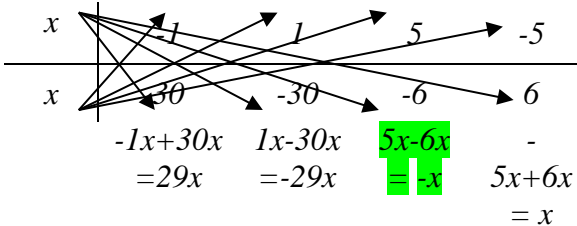
Mental Strategy	Sequence of Steps Strategy
The first term of the trinomial, $3x^2$, came from $3x \times x$. To obtain 6 from two numbers, both will be positive or both negative. With the middle term negative, the factor pairs of 6 are -1 and -6, or -2 and -3. $3x$ -1 -2 x -6 -3 *-1x-18x =- 19x* *-2x-9x = - 11x* This indicates the correct option is: $3x^2 - 11x + 6 = (3x - 2)(x - 3)$	So for the trinomial being factorised $3x^2 - 11x + 6$; $ac = 3 \times 6 = 18$ and $b = -11$. Two numbers that multiply to 18 and add to -11 are -2 and -9 (-2 x -9 = 18, -2 + -9 = -11) This gives us the breakdown of the middle term $-11x$ into two separate terms $-2x$ and $-9x$. Now the trinomial becomes: $$3x^2 - 11x + 6$$ $$= 3x^2 - 2x - 9x + 6$$ $$= x(3x - 2) - 3(3x - 2)$$ $$= (3x - 2)(x - 3)$$

Example: Factorise the trinomial $3x^2 - 3x - 90$

Looking at this trinomial, you will notice that there is a common factor of 3 in each term, so the first step in this factorisation is taking out the common factor. This makes the remaining factorising much easier
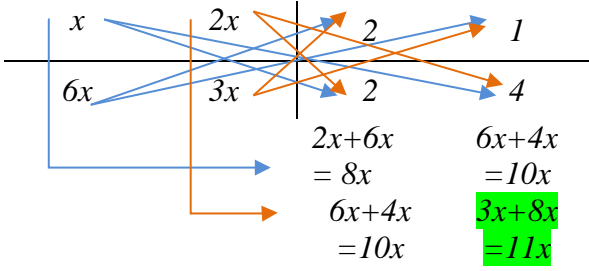
$$3x^2 - 3x - 90 = 3(x^2 - x - 30)$$

Now the trinomial: $x^2 - x - 30$ is to be factorised

Mental Strategy	Sequence of Steps Strategy
The first term of the trinomial, x^2, came from $x \times x$. To obtain -30 from two numbers, one will be positive and one will be negative. The factor pairs of -30 are -1 and 30, 1 and -30, 5 and -6 or -5 and 6. x -1 1 5 -5 x 30 -30 -6 6 *-1x+30x =29x* *1x-30x =-29x* *5x-6x = -x* *5x+6x = x* This indicates the correct option is: $x^2 - x - 30 = (x + 5)(x - 6)$	So for the trinomial being factorised $x^2 - x - 30$; $ac = 1 \times -30 = -30$ and $b = -1$. Two numbers that multiply to -30 and add to -1 are 5 and -6 (5 x -6 = -30, 5 + -6 = -1) This gives us the breakdown of the middle term $-1x$ into two separate terms $5x$ and $-6x$. Now the trinomial becomes: $$x^2 - x - 30$$ $$= x^2 + 5x - 6x - 30$$ $$= x(x + 5) - 6(x + 5)$$ $$= (x + 5)(x - 6)$$

The final result is $3x^2 - 3x - 90 = 3(x^2 - x - 30) = 3(x + 5)(x - 6)$

Example: Factorise the trinomial $6x^2 + 11x + 4$

Mental Strategy	Sequence of Steps Strategy
The first term of the trinomial, $6x^2$, came from $x \times 6x$ or $2x \times 3x$. To obtain 4 from two numbers, both will be positive. The factor pairs of 4 are 1 and 4, or 2 and 2. **This time there is two options for the first term.** This indicates the correct option is: $$6x^2 + 11x + 4 = (2x+1)(3x+4)$$	So for the trinomial being factorised $6x^2 + 11x + 4$; $ac = 6 \times 4 = 24$ and $b = 11$. Two numbers that multiply to 24 and add to 11 are 3 and 8 ($3 \times 8 = 24$, $3 + 8 = 11$) This gives us the breakdown of the middle term $11x$ into two separate terms $3x$ and $8x$. Now the trinomial becomes: $$6x^2 + 11x + 4$$ $$= 6x^2 + 3x + 8x + 4$$ $$= 3x(2x+1) + 4(2x+1)$$ $$= (3x+4)(2x+1)$$

Difference of squares

A difference of squares is written as the identity $a^2 - b^2 = (a+b)(a-b)$

You may remember seeing this, reversed around, as a useful identity for expanding.

When do you know if you have a Difference of Squares?

The example below is a Difference of Squares

$$x^2 \quad - \quad 4$$

x^2 is x squared

Difference
(subtract)

4 is 2 squared (2^2)
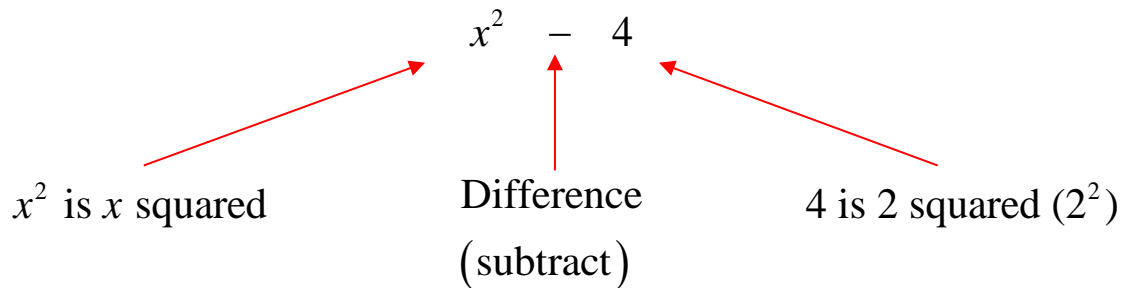
Note: There is no middle term

Example: Factorise $x^2 - 4$

$$x^2 - 4$$
$$= x^2 - 2^2$$
$$= (x-2)(x+2)$$

Example: Factorise $x^2 + 16$

This is not a difference of squares

Example: Factorise $x^2 - 225$

$$x^2 - 225$$
$$= x^2 - 15^2$$
$$= (x+15)(x-15)$$

Example: Factorise $3x^2 - 75$

$3x^2 - 75$ • Common factor of 3
$$= 3(x^2 - 25)$$
$$= 3(x^2 - 5^2)$$
$$= 3(x-5)(x+5)$$

Example: Factorise $4x^2 - 49$

$$4x^2 - 49$$
$$= (2x)^2 - 7^2$$
$$= (2x-7)(2x+7)$$

The question below uses our knowledge of square and square roots.

Example: Factorise $x^2 - 5$

$$x^2 - 5$$
$$= x^2 - \left(\sqrt{5}\right)^2 \qquad \bullet 5 \text{ can be rewritten as } \left(\sqrt{5}\right)^2$$
$$= (x - \sqrt{5})(x + \sqrt{5})$$

Perfect squares

The identities for perfect squares are:

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$\text{and } a^2 - 2ab + b^2 = (a-b)^2$$

You may remember seeing these, reversed around, as useful identities for expanding. However, for factorising you need to be able to see the how the middle term relates to *2ab*.

Look at the trinomial. If the first and last terms are perfect squares, then examine the middle more closely. If the middle term is *2ab* where *a* is from the first term and *b* is from the last term, then it is a perfect square. Remember you can always factorise as a trinomial, you will get the same result.

Example: $\underset{x^2}{x^2} + \underset{2\times3\times x}{6x} + \underset{3^2}{9} = (x+3)^2$

Example: $\underset{x^2}{x^2} - \underset{2\times^-5\times x}{10x} + \underset{5^2}{25} = (x-5)^2$

Example: $\underset{(2x)^2}{4x^2} + \underset{2\times1\times2x}{4x} + \underset{1^2}{1} = (2x+1)^2$

Example: $\underset{(2x)^2}{4x^2} - \underset{2\times^-5\times2x}{20x} + \underset{5^2}{25} = (2x-5)^2$

Sum and difference of cubes

Two other identities worth knowing are the sum and difference of cubes. The appearance of these is below:

Sum of Cubes	Difference of Cubes
Identity: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$	Identity: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
Examples: (i) $x^3 + 8$ $= \underset{a^3}{x^3} + \underset{b^3}{2^3}$ $= \underset{a+b}{(x+2)}\underset{a^2-ab+b^2}{(x^2 - 2x + 4)}$ (ii) $108 + 4x^3$ $\bullet$ common factor of 4 $= 4(27 + x^3)$ $= 4(3^3 + x^3)$ $= 4(3 + x)(9 + 3x + x^2)$ (iii) $64x^3 + 1$ $= (4x)^3 + 1^3$ $= (4x+1)(16x^2 - 4x + 1)$	Examples: (i) $x^3 - 125$ $= \underset{a^3}{x^3} - \underset{b^3}{5^3}$ $= \underset{a+b}{(x-5)}\underset{a^2-ab+b^2}{(x^2 + 5x + 25)}$ (ii) $-1 + 8x^3$ $= 8x^3 - 1$ $= (2x)^3 - 1^3$ $= (2x-1)(4x^2 + 2x + 1)$ (iii) $24x^3 - 81y^3$ $\bullet$ common factor of 3 $= 3\left[8x^3 - 27y^3\right]$ $= 3\left[(2x)^3 - (3y)^3\right]$ $= 3(2x - 3y)(4x^2 + 6xy + 9y^2)$

Factorising a sum or difference of cubes results in the product of a binomial factor and a trinomial factor, which can be written as $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$. The trinomial factor will not factorise further.

The video below covers some examples similar to those above. Click on the image below to start downloading the video. Depending on your internet connection, this may take a minute or two.

Video 'Factorisation'

Activity

1. Factorise the following.

(a) $8a + 12$ (b) $6x - 12$

(c) $12p - 36$ (d) $7a - 21b$

(e) $bg + 2bf$ (f) $14a + 24b$

(g) $a^2 + 2a$ (h) $6x - 6x^2$

(i) $12p^2 - 6p$ (j) $14a - 21a^2$

(k) $3g^2 + 2g$ (l) $12a - 20a^2$

(m) $8a^2 + 12ab$ (n) $9ax^2 - 12a^2x$

(o) $a^4b^3 - a^3b^3$ (p) $7a^2b - 49ab^2$

(q) $^-2g - 6g^2$ (r) $^-4x^3 - 24x$

2. Factorise the following.(Grouping factors)

(a) $a(x+y) + 2(x+y)$ (b) $3(x+y)^2 + 2(x+y)$

(c) $a(x-y) - x + y$ (d) $3a(x-y) - 2(y-x)$

(e) $4a^4(y-3) - 6a^3(y-3)$ (f) $3(p+q)^2 - 6(p+q)$

(g) $3a + ab + 3c + bc$ (h) $a^2 + 3a - ac - 3c$

(i) $a^3 - 3a^2 + ab - 3b$ (j) $x^2y^2 + 1 - x^2y - y$

(k) $2xy^2 + 2 + y^2 + 4x$ (l) $12x^3 + x^2y - 4x^3y - 3x^2$

(m) $ax + ay + bx + by$ (n) $4x^2 + x - 12ax - 3a$

(o) $6x^3 - 9x^2 - 2x + 3$ (p) $2a^2b - 2a - b + ab^2$

3. Factorise the following.(Common factor - exponent)

(a) $4m^4 - 2m^2$ (b) $x^{n+2} + x^{n-2}$

(c) $e^{x+1} + e^x$ (d) $e^{2x} - e^{-2x}$

(e) $e^{4m} + e^{2m}$

4. The following questions are partially factorised, complete the missing factor.

(a) $x^2 + 3x + 2 = (x+2)(x+...)$ (b) $x^2 + 5x + 6 = (x+2)(x+...)$

(c) $x^2 + 7x + 12 = (x+3)(x+...)$ (d) $x^2 - 3x + 2 = (x-2)(x-...)$

(e) $x^2 - 7x + 10 = (x-5)(x-...)$ (f) $x^2 - 2x - 8 = (x+2)(x-...)$

(g) $x^2 + 2x - 35 = (x+7)(x-...)$ (h) $x^2 + 4x - 12 = (x-2)(x+...)$

(i) $x^2 - 2x - 15 = (x+3)(... ... ...)$ (j) $x^2 - x - 20 = (x+4)(... ... ...)$

(k) $x^2 - 10x + 9 = (x-1)(... ... ...)$ (l) $x^2 + 15x + 56 = (x+7)(... ... ...)$

5. Factorise these trinomials.

(a) $a^2 + 7a + 6$ (b) $p^2 + 9p + 20$

(c) $a^2 - 10a + 24$ (d) $a^2 + 11a + 24$

(e) $x^2 - 15x + 36$ (f) $x^2 + 13x + 36$

(g) $x^2 + 16x - 36$ (h) $2x^2 + 5x - 3$

(i) $3x^2 - 11x + 6$ (j) $4x^2 + 25x - 21$

(k) $6x^2 - x - 1$ (l) $4a^2 - 12a + 5$

(m) $5y^2 + 13y - 6$ (n) $3b^2 + 14b - 5$

(o) $10x^2 + 9x - 9$ (p) $30m^2 - 37m + 10$

(q) $6a^2 - 47a + 77$ (r) $8x^2 + 2x - 15$

6. Factorise the following by using the difference of squares method.

(a) $b^2 - 49$ (b) $81 - x^2$

(c) $4b^2 - 49$ (d) $9x^2 - 4y^2$

(e) $4a^2 - 25b^2$ (f) $(x+1)^2 - 49$

7. The following questions can be factorised by the perfect square method. Factorise the following by using the perfect square method or alternatively, factorise as trinomials.

(a) $a^2 + 14a + 49$ (b) $x^2 - 12x + 36$

(c) $b^2 - 22b + 121$ (d) $4x^2 + 4x + 1$

(e) $25a^2 - 20a + 4$ (f) $4x^2 + 28x + 49$

8. Factorise the following by using the sum or difference of cubes method.

(a) $b^3 - 8$ (b) $64 - x^3$

(c) $8b^3 - 125$ (d) $x^3 - 8y^3$

(e) $27x^3 - 125b^3$ (f) $64x^3 - 1000$

9. Factorise the questions below, one or more of the skills above is required.

(a) $6x^2 - 33x + 36$ (b) $48x^2 - 3$

(c) $x^4 + 2x^2 - 3$ (d) $(x-y)^2 - (x+y)^2$

(e) $x^2 + 6x + 9 - y^2$ (f) $x^3 + 3x^2 - xb^2 - 3b^2$

(g) $(x+y)^2 + 2(x+y) + 1$ (h) $a^2 - 9 - 4ac + 12c$

(i) $5m^3 - m^2 - 20m + 4$ (j) $x^6 - 1$

(k) $4x^2y^2 - x^2 - 4xy - 4y^2$ (l) $2(2a+1)^2 - 8(a-2)^2$

The answers to the activity questions are found in the pdf named:

'Answers to Activity Questions (PCA)'

Topic 3: Algebraic fractions

Simplifying fractions can only be performed when there is a common factor both in the numerator and denominator. Remember factors are numbers or terms that multiply.

For example: In the fraction $\dfrac{5ab}{20ac}$ there is a common factor of 5 and a in both the numerator and denominator.

$$\frac{5ab}{20ac}$$
$$=\frac{\cancel{5}^{1}\times\cancel{a}^{1}\times b}{\cancel{20}^{4}\times\cancel{a}^{1}\times c}$$
$$=\frac{b}{4c}$$

In this example, $\dfrac{x^2+1}{x-3}$ there is no common factor in both numerator and denominator so it cannot be simplified. However in some situations, the division could be performed.

Simplifying

$\dfrac{3t-12}{3t}$ $=\dfrac{\cancel{3}(t-4)}{\cancel{3}t}$ $=\dfrac{t-4}{t}$	$\dfrac{m^2n-4mn}{m-4}$ $=\dfrac{mn\cancel{(m-4)}}{\cancel{m-4}}$ $=\dfrac{mn}{1}=mn$
$\dfrac{4m^2n-16n}{8mn}$ $=\dfrac{\cancel{4}\,\cancel{n}(m^2-4)}{\cancel{8}^2\,m\cancel{n}}$ $=\dfrac{m^2-4}{2m}$ or $\dfrac{(m-2)(m+2)}{2m}$	$\dfrac{6a^2bc+3ab^2c-9abc^2}{15abc}$ $=\dfrac{\cancel{3}\,\cancel{abc}\,(2a+b-3c)}{^{5}\cancel{15}\,\cancel{abc}}$ $=\dfrac{2a+b-3c}{5}$

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$\dfrac{3-x}{x-3}$ You may recall $3 - x = -(x-3)$ $= \dfrac{-(x-3)^{1}}{(x-3)^{1}}$ $= {}^{-}1$	$\dfrac{x^2+5x+6}{x+3}$ $= \dfrac{(x+2)\,(x+3)}{x+3}$ $= x+2$
$\dfrac{x^2-3x-10}{5-x}$ $= \dfrac{(x+2)(x-5)}{-(x-5)}$ $= -(x+2)$ or $-x-2$	$\dfrac{x^2+4x+3}{x^2+2x-3}$ factorise each trinomial $= \dfrac{(x+1)\,(x+3)}{(x-1)\,(x+3)}$ $= \dfrac{x+1}{x-1}$
$\dfrac{x^2-4}{x^2-2x}$ $= \dfrac{(x-2)\,(x+2)}{x\,(x-2)}$ $= \dfrac{x+2}{x}$	$\dfrac{2t^2+7t-4}{6t^2+5t-4}$ $= \dfrac{(2t-1)\,(t+4)}{(2t-1)\,(3t+4)}$ $= \dfrac{t+4}{3t+4}$

Adding and subtracting

When adding or subtracting fractions, it is essential that each fraction is expressed with a common denominator. This could be summarised by this expression:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

or

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

However, there are times when a smaller common denominator is possible, for example:

$$\frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2} \quad \text{or} \quad \frac{2}{m} + \frac{5}{2m} = \frac{4+5}{2m} = \frac{9}{2m}$$

Examples:

$\dfrac{1}{x}+\dfrac{3}{2x}-\dfrac{5}{3x}$ • $6x$ is the L.C.D. $=\dfrac{6+9-10}{6x}$ $=\dfrac{5}{6x}$	$\dfrac{2a}{x}+\dfrac{3}{x^2}$ • x^2 is the L.C.D. $=\dfrac{2ax+3}{x^2}$
$\dfrac{x+1}{5}-\dfrac{x-2}{3}$ •15 is the L.C.D. $=\dfrac{3(x+1)-5(x-2)}{15}$ • put numerators in brackets $=\dfrac{3x+3-5x+10}{15}$ • expand and collect like terms $=\dfrac{-2x+13}{15}$ or $\dfrac{13-2x}{15}$	$\dfrac{3+x}{x^2}+\dfrac{1-4x}{3x}$ •$3x^2$ is the C.D. $=\dfrac{3(3+x)+x(1-4x)}{3x^2}$ $=\dfrac{9+3x+x-4x^2}{3x^2}$ $=\dfrac{9+4x-4x^2}{3x^2}$ $9+4x-4x^2$ is a trinomial that will not factorise.
$2y-\dfrac{y-4}{5}$ $=\dfrac{10y-(y-4)}{5}$ if the numerator contains more than 1 term - put numerator in brackets $=\dfrac{10y-y+4}{5}$ $=\dfrac{9y+4}{5}$	$\dfrac{4}{2x-3}-\dfrac{1}{x+7}$ $=\dfrac{4(x+7)-1(2x-3)}{(2x-3)(x+7)}$ $=\dfrac{4x+28-2x+3}{(2x-3)(x+7)}$ $=\dfrac{2x+31}{(2x-3)(x+7)}$
$\dfrac{x-4}{x^2-9}+\dfrac{4}{x+3}$ •x^2-9 is a difference of squares $=(x+3)(x-3)$ $=\dfrac{x-4+4(x-3)}{(x+3)(x-3)}$ $=\dfrac{x-4+4x-12}{(x+3)(x-3)}$ $=\dfrac{5x-16}{(x+3)(x-3)}\ or\ \dfrac{5x-16}{x^2-9}$	$\dfrac{x+3}{x^2-x-2}-\dfrac{x-1}{x^2+x-6}$ • Factorise trinomials $=\dfrac{x+3}{(x-2)(x+1)}-\dfrac{x-1}{(x+3)(x-2)}$ $=\dfrac{(x+3)(x+3)-(x-1)(x+1)}{(x+1)(x+3)(x-2)}$ $=\dfrac{x^2+6x+9-(x^2-1)}{(x+1)(x+3)(x-2)}$ $=\dfrac{x^2+6x+9-x^2+1}{(x+1)(x+3)(x-2)}$ $=\dfrac{6x+10}{(x+1)(x+3)(x-2)}$ or $\dfrac{2(3x+5)}{(x+1)(x+3)(x-2)}$

$$\frac{3x-3}{x^2-6x+5} - \frac{4}{5-x}$$

$$= \frac{3\cancel{(x-1)}}{\cancel{(x-1)}(x-5)} - \frac{4}{-(x-5)}$$

$$= \frac{3}{x-5} + \frac{4}{x-5}$$

$$= \frac{7}{x-5}$$

$$\frac{4x}{x^3-x^2-30} - \frac{x-1}{x^2+5x} + \frac{3-2x}{x-6}$$

$$= \frac{4x}{x(x+5)(x-6)} - \frac{x-1}{x(x+5)} + \frac{3-2x}{x-6}$$

$$= \frac{4x-(x-1)(x-6)+(3-2x)x(x+5)}{x(x+5)(x-6)}$$

$$= \frac{4x-(x^2-7x+6)+(-2x^3-7x^2+15x)}{x(x+5)(x-6)}$$

$$= \frac{4x-x^2+7x-6-2x^3-7x^2+15x}{x(x+5)(x-6)}$$

$$= \frac{-2x^3-8x^2+26x-6}{x(x+5)(x-6)} \quad \text{or} \quad \frac{-2(x^3+4x^2-13x+3)}{x(x+5)(x-6)}$$

Multiplying and dividing

When multiplying and dividing fractions, just follow the same logic as for numbers. Simplifying before or after multiplying.

Multiplying	Dividing
$\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$	$\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$

Examples:

Multiplying	Dividing
$\dfrac{12a^2b}{4c} \times \dfrac{5ac}{b^3}$ $= \dfrac{\cancel{12}^3 a^2 \cancel{b}}{\cancel{4}^1 \cancel{c}} \times \dfrac{5a\cancel{c}}{b^{3}\ ^{b^2}}$ $= \dfrac{3a^3}{b^2}$ **or** $\begin{aligned} &\dfrac{12a^2b}{4c} \times \dfrac{5ac}{b^3} \\ &= \dfrac{12a^3b^1c^1}{4b^3c^1} \\ &= 3a^3b^{1-3}c^{1-1} \\ &= 3a^3b^{-2}c^0 \\ &= \dfrac{3a^3}{b^2} \end{aligned}$	$\dfrac{3}{ab} \div \dfrac{4a}{b}$ $= \dfrac{3}{a\cancel{b}} \times \dfrac{\cancel{b}}{4a}$ $= \dfrac{3}{4a^2}$
$\dfrac{x+3}{4} \times \dfrac{5a}{x+3}$ $= \dfrac{5a\,\cancel{(x+3)}}{4\,\cancel{(x+3)}}$ $= \dfrac{5a}{4}$	$\dfrac{15a^2b^4c}{9ac} \div 5b^2$ $= \dfrac{15a^2b^4c}{9ac} \div \dfrac{5b^2}{1}$ $= \dfrac{15a^2b^4c}{9ac} \times \dfrac{1}{5b^2}$ $= \dfrac{\cancel{15}^1 a^2 b^4 \cancel{c}}{\cancel{45}^3 ab^2 \cancel{c}}$ $= \dfrac{ab^2}{3}$
$\dfrac{2x^2+x-6}{x^2+3x} \times \dfrac{x^2+6x+9}{x^2+5x+6}$ $= \dfrac{(2x-3)\,\cancel{(x+2)}}{x\,\cancel{(x+3)}} \times \dfrac{\cancel{(x+3)}^2}{\cancel{(x+2)}\,\cancel{(x+3)}}$ $= \dfrac{2x-3}{x}$	$\dfrac{y^2-y}{y^2-1} \div \dfrac{4y^2+8y+4}{y^2-y-2}$ $= \dfrac{y(y-1)}{(y-1)(y+1)} \times \dfrac{y^2-y-2}{4y^2+8y+4}$ $= \dfrac{y\,\cancel{(y-1)}}{\cancel{(y-1)}\,(y+1)} \times \dfrac{(y-2)\,\cancel{(y+1)}}{4(y+1)^2}$ $= \dfrac{y(y-2)}{4(y+1)^2}$
$\dfrac{x^2+4x}{2x^2-12x+16} \times \dfrac{x^2-x-2}{x^2+5x+4}$ $= \dfrac{x(x+4)}{2(x^2-6x+8)} \times \dfrac{(x-2)(x+1)}{(x+4)(x+1)}$ $= \dfrac{x\,\cancel{(x+4)}}{2(x-4)\,\cancel{(x-2)}} \times \dfrac{\cancel{(x-2)}\,\cancel{(x+1)}}{\cancel{(x+4)}\,\cancel{(x+1)}}$ $= \dfrac{x}{2x-8}$	$(x^2+8x+15) \div \dfrac{x^2}{x^3-x^2-2x}$ $= \dfrac{x^2+8x+15}{1} \times \dfrac{x^3-x^2-2x}{x^2}$ $= \dfrac{(x+3)(x+5)}{1} \times \dfrac{\cancel{x}(x-2)(x+1)}{x^{\cancel{2}}}$ $= \dfrac{(x+1)(x-2)(x+3)(x+5)}{x}$

Solving complex fractions

Complex fractions are fractions where the numerator and denominator are written as fractions. The example below is considered complex because of the numerator.

$$\frac{\dfrac{1}{a}-a}{1-a}$$

There are many methods to solve this, but generally it is helpful to rewrite this from fraction form into a form containing a ÷ sign.

$$\left(\frac{1}{a}-a\right)\div(1-a)$$

Now the numerator can be written with a common denominator. The division (÷) can be changed into a multiplication provided that it is a single fraction.

$$\left(\frac{1-a^2}{a}\right)\times\frac{1}{(1-a)}$$

This can now be simplifying using all the algebra knowledge available.

$$\left(\frac{1-a^2}{a}\right)\times\frac{1}{(1-a)}$$
$$=\frac{\cancel{(1-a)}(1+a)}{a\cancel{(1-a)}}$$
$$=\frac{1+a}{a}$$

Example:

$$\frac{\dfrac{1}{x+3}-1}{1+\dfrac{1}{x+3}}$$
$$=\left(\frac{1}{x+3}-1\right)\div\left(1+\frac{1}{x+3}\right)$$
$$=\left(\frac{1-(x+3)}{x+3}\right)\div\left(\frac{(x+3)+1}{x+3}\right)$$
$$=\left(\frac{-2-x}{x+3}\right)\div\left(\frac{x+4}{x+3}\right)$$
$$=\frac{-2-x}{\cancel{x+3}}\times\frac{\cancel{x+3}}{x+4}$$
$$=\frac{-2-x}{x+4}$$

Example:

$$\frac{3 - \dfrac{1}{2x}}{x + \dfrac{x}{x+2}}$$

$$= \left(\frac{6x-1}{2x}\right) \div \left(\frac{x(x+2)+x}{x+2}\right)$$

$$= \frac{6x-1}{2x} \times \frac{x+2}{x^2+3x}$$

$$= \frac{(6x-1)(x+2)}{2x^2(x+3)}$$

The video below covers some examples similar to those above. Click on the image below to start downloading the video. Depending on your internet connection, this may take a minute or two.

Video 'Algebraic Fractions'

Activity

1. Simplify the following

(a) $\dfrac{6a^2bc}{3ab^2}$

(b) $\dfrac{12ab}{a^2c}$

(c) $\dfrac{10-2t}{5t}$

(d) $\dfrac{a^2b-ab^2}{a-b}$

(e) $\dfrac{3x^2y-12y}{x-2}$

(f) $\dfrac{4x^2yz-6xy^2z+2xyz^2}{2xy}$

(g) $\dfrac{9-3x}{3x-9}$

(h) $\dfrac{x^2+2x-15}{3-x}$

(i) $\dfrac{x^2+2x-15}{2x^2+7x-15}$

(j) $\dfrac{4t^2-36}{t^2-3t}$

(k) $\dfrac{x^2+10x+25}{x^2-25}$

(l) $\dfrac{6+7x-3x^2}{3x^2-x-2}$

2. Perform the addition or subtraction. Simplify as much as possible.

(a) $\dfrac{4}{m}+\dfrac{2}{n}$

(b) $\dfrac{n}{m}+\dfrac{2}{3m}$

(c) $\dfrac{2}{x}-\dfrac{1}{4x}-\dfrac{3}{5x}$

(d) $\dfrac{1}{x^2}+\dfrac{2}{4x}+\dfrac{1}{5}$

(e) $\dfrac{3}{x+2}+\dfrac{x}{x-2}$

(f) $\dfrac{x-2}{4}-\dfrac{x+1}{3}$

(g) $\dfrac{2(x+2)}{5}+\dfrac{3(4-x)}{4}$

(h) $y^2-\dfrac{y+2}{y}$

(i) $\dfrac{2-x}{x^2-16}+\dfrac{3}{x+4}$

(j) $\dfrac{x-3}{x^2-4x-5}-\dfrac{2(2x+1)}{2x^2-11x+5}$

(k) $\dfrac{1}{x^3-9x}+\dfrac{4}{x-3}-\dfrac{5}{x+3}$

(l) $\dfrac{x^2-5x}{2x^2+7x+3}+\dfrac{4x^2-1}{x^2-2x-15}$

3. Simplify the following

(a) $\dfrac{7a^2b}{5c}\times\dfrac{2ac}{b^2}$

(b) $\dfrac{4a^2b}{3c}\div\dfrac{2a}{3b}$

(c) $\dfrac{3x+9}{5}\times\dfrac{10}{x+3}$

(d) $\dfrac{16x^2y^2z^4}{5x}\div\dfrac{x^2}{2}$

(e) $\dfrac{x^2-4x}{x^2-4}\times\dfrac{x^2+4x+4}{x^2-2x-8}$

(f) $\dfrac{x^2+3x+2}{x^2-7x+12}\div\dfrac{2x^2+8x+8}{x^2+7x+6}$

4. Simplify the following

(a) $\dfrac{2-\dfrac{1}{x}}{\dfrac{2}{x}-5}$

(b) $\dfrac{\dfrac{3}{x-2}+1}{5-\dfrac{1}{x+2}}$

The answers to the activity questions are found in the pdf named:

'Answers to Activity Questions (PCA)'

Topic 4: Binomial expansions

Let's consider the following expansions using a simple binomial. The expansions are written to clearly show the coefficients of each term.

$$(a+b)^0 = 1$$

(Any expression to the power of 0 gives 1)

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = (a+b)(a+b) = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = (a+b)(a+b)(a+b) = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b) = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

etc.

By observation, you notice that the first term a of the binomial expands to the powers shown (which gets smaller with each term).

$$(a+b)^5 = 1\underline{a}^5 + 5\underline{a}^4b + 10\underline{a}^3b^2 + 10\underline{a}^2b^3 + 5\underline{a}^1b^4 + 1b^5 \text{ or } (1\underline{a}^0b^5)$$

exponent of $\underline{a}$ goes 5 4 3 2 1 0 ($a^0=1$)

Also, the second term b of the binomial expands to the powers shown (which increases with each term).

$$(a+b)^5 = 1a^5\underline{b}^0 + 5a^4\underline{b}^1 + 10a^3\underline{b}^2 + 10a^2\underline{b}^3 + 5a\underline{b}^4 + 1\underline{b}^5$$

exponent of $\underline{b}$ goes 0 1 2 3 4 5

Also, in the expansion, you notice that the sum of the exponents in each term is always equal to the value of the exponent the binomial expanded to.

For example:

$$(a+b)^5 = 1a^5\underline{b}^0 \; ^{(5+0=5)} + 5a^4\underline{b}^1 \; ^{(4+1=5)} + 10a^3\underline{b}^2 \; ^{(3+2=5)} + 10a^2\underline{b}^3 \; ^{(2+3=5)} + 5a\underline{b}^4 \; ^{(1+4=5)} + 1a^0\underline{b}^5 \; ^{(0+5=5)}$$

Now continuing this pattern, we know that;

$$(a+b)^6 = C_1a^6b^0 + C_2a^5b^1 + C_3a^4b^2 + C_4a^3b^3 + C_5a^2b^4 + C_6a^1b^5 + C_7a^0b^6$$

where C_1 to C_7 are coefficients of each term.

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Let's look at the coefficients used so far;

$$
\begin{array}{c}
1 \\
1\ \ 1 \\
1\ \ 2\ \ 1 \\
1\ \ 3\ \ 3\ \ 1 \\
1\ \ 4\ \ 6\ \ 4\ \ 1 \\
1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1
\end{array}
$$

The coefficients form a triangular structure. This is called **Pascal's Triangle**. The first and last number in each line of the triangle is 1. Each new line in the triangle is formed from the previous one.

$$
\begin{array}{c}
1 \\
1\ \ 1 \\
\vee \\
1\ \ 2\ \ 1 \\
\vee\vee \\
1\ \ 3\ \ 3\ \ 1 \\
\vee\ \vee\ \vee \\
1\ \ 4\ \ 6\ \ 4\ \ 1 \\
\vee\ \ \vee\ \ \vee\ \ \vee \\
1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \\
\vee\ \vee\ \ \vee\ \ \vee\ \vee \\
1\ \ 6\ \ 15\ \ 20\ \ 15\ \ 6\ \ 1
\end{array}
$$

Adding a new row

Now the expansion can be completed:

$$(a + b)^6 = 1a^6b^0 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1a^0b^6$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Continuing on to $(a + b)^7$

Step 1: Obtain the next line of Pascal's Triangle.

$$
\begin{array}{c}
1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \\
\vee\ \ \vee\ \ \vee\ \ \vee\ \ \vee \\
1\ \ 6\ \ 15\ \ 20\ \ 15\ \ 6\ \ 1 \\
\vee\ \vee\ \vee\ \ \vee\ \ \vee\vee \\
1\ \ 7\ \ 21\ \ 35\ \ 35\ \ 21\ \ 7\ \ 1
\end{array}
$$

Adding a new row

$$(a + b)^7 = 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

Step 2: Put in the powers from the first term of the binomial.

$$(a + b)^7 = 1a^7 \quad 7a^6 \quad 21a^5 \quad 35a^4 \quad 35a^3 \quad 21a^2 \quad 7a^1 \quad 1a^0$$

Step 3: Put in the powers from the second term of the binomial.

$$(a + b)^7 = 1a^7b^0 \quad 7a^6b^1 \quad 21a^5b^2 \quad 35a^4b^3 \quad 35a^3b^4 \quad 21a^2b^5 \quad 7a^1b^6 \quad 1a^0b^7$$

Step 4: Put in addition signs and simplify:

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

If the binomial is more complex, this method will still work; however, more care with simplifying is required.

Example: Expand $(2a+5)^5$ using Pascal's Triangle

Step 1: Obtain the appropriate line of Pascal's Triangle.
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$(2a+5)^5 = 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

Step 2: Put in the powers from the first term of the binomial. The first term should be put in brackets so that the coefficient of a is also raised to the exponent. For example: $(2a)^5 = 32a^5$
$$(2a+5)^5 = 1(2a)^5 \quad 5(2a)^4 \quad 10(2a)^3 \quad 10(2a)^2 \quad 5(2a)^1 \quad 1(2a)^0$$

Step 3: Put in the powers from the second term of the binomial.

$$(2a+5)^5 = 1(2a)^5(5)^0 \quad 5(2a)^4(5)^1 \quad 10(2a)^3(5)^2 \quad 10(2a)^2(5)^3 \quad 5(2a)^1(5)^4 \quad 1(2a)^0(5)^5$$

Step 4: Put in addition signs and simplify:
$$(2a+5)^5 = (2a)^5 + 5(2a)^4(5)^1 + 10(2a)^3(5)^2 + 10(2a)^2(5)^3 + 5(2a)^1(5)^4 + (5)^5$$
$$= 32a^5 + (5 \times 16 \times 5)a^4 + (10 \times 8 \times 25)a^3 + (10 \times 4 \times 125)a^2 + (5 \times 2 \times 625)a + 3125$$
$$= 32a^5 + 400a^4 + 2000a^3 + 5000a^2 + 6250a + 3125$$

In this example, the binomial consists of a subtraction. The process outlined is the same but even more care taken when simplifying.

Example: Expand $(x-3)^4$ using Pascal's Triangle

Step 1: Obtain the appropriate line of Pascal's Triangle.
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$(x-3)^4 = 1 \quad 4 \quad 6 \quad 4 \quad 1$$

Step 2: Put in the powers from the first term of the binomial.
$$(x-3)^4 = 1(x)^4 \quad 4(x)^3 \quad 6(x)^2 \quad 4(x)^1 \quad 1(x)^0$$

Step 3: Put in the powers from the second term of the binomial.
$$(x-3)^4 = 1(x)^4(-3)^0 \quad 4(x)^3(-3)^1 \quad 6(x)^2(-3)^2 \quad 4(x)^1(-3)^3 \quad 1(x)^0(-3)^4$$

Step 4: Put in addition signs and simplify:
$$(x-3)^4 = 1(x)^4 + 4(x)^3 (-3)^1 + 6(x)^2 (-3)^2 + 4(x)^1 (-3)^3 + 1(-3)^4$$
$$= x^4 + (4 \times -3) x^3 + (6 \times 9) x^2 + (4 \times -27) x + 81$$
$$= x^4 - 12x^3 + 54x^2 - 108x + 81$$

For example: Expand $(x^2 - 3)^5$ using Pascal's Triangle

Step 1: Obtain the appropriate line of Pascal's Triangle.
$$\begin{array}{cccccc} 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$
$$(x^2 - 3)^5 = \begin{array}{cccccc} 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

Step 2: Put in the powers from the first term of the binomial.
$$(x^2 - 3)^5 = 1(x^2)^5 \quad 5(x^2)^4 \quad 10(x^2)^3 \quad 10(x^2)^2 \quad 5(x^2)^1 \quad 1(x^2)^0$$

Step 3: Put in the powers from the second term of the binomial.
$$(x^2 - 3)^5 = 1(x^2)^5 (-3)^0 \quad 5(x^2)^4 (-3)^1 \quad 10(x^2)^3 (-3)^2 \quad 10(x^2)^2 (-3)^3 \quad 5(x^2)^1 (-3)^4 \quad 1(x^2)^0 (-3)^5$$

Step 4: Put in addition signs and simplify:
$$(x^2 - 3)^5 = 1(x^2)^5 (-3)^0 + 5(x^2)^4 (-3)^1 + 10(x^2)^3 (-3)^2 + 10(x^2)^2 (-3)^3 + 5(x^2)^1 (-3)^4 + 1(x^2)^0 (-3)^5$$
$$= x^{10} + (5 \times -3) x^8 + (10 \times 9) x^6 + (10 \times -27) x^4 + (5 \times 81) x^2 + (1 \times -243)$$
$$= x^{10} - 15x^8 + 90x^6 - 270x^4 + 405x^2 - 243$$

If the question $(a+b)^{12}$ was given, writing Pascal's triangle to give the coefficients required would be long and tedious. Another way of getting the coefficients is to use the mathematical idea of a **combination**.

In a combination, the order objects are drawn is not important. In lotto, you need 6 winning numbers out of 45 to win. The order they are drawn is not important. This is an example of a combination.

From a class of 5 students, 2 students are selected to represent the class at a forum. If the 5 students are represented by the letters A, B, C, D and E, then all the combinations containing two students are: AB, AC, AD, AE, BC, BD, BE, CD, CE and DE, giving 10 choices. Notice BA was not listed because it is the same as AB, and order is not important.

The number of combinations when r objects are selected from n objects is given by the formula:
$$\boxed{^nC_r = \frac{n!}{r! \times (n-r)!}}$$

^nC_r is shorthand for when r objects are selected from n objects

$n!$ means n factorial. The factorial of a number is the product of all the natural numbers up to and including the number. There is a special case that does not fit this definition; $0! = 1$.

5! means 5 x 4 x 3 x 2 x 1 = 120

n! means n x (n-1) x (n-2) … x 3 x 2 x 1

For the example above: $n=5$ and $r=2$

$$^{n}C_{r} = \frac{n!}{r! \times (n-r)!}$$

$$= \frac{5!}{2! \times (5-2)!}$$

$$= \frac{5 \times \cancel{4}^{2} \times \overline{3 \times 2 \times 1}}{\cancel{2} \times 1 \times \overline{3 \times 2 \times 1}}$$

$$= \frac{10}{1}$$

$$= 10$$

or

$$^{n}C_{r} = \frac{n!}{r! \times (n-r)!}$$

$$= \frac{5!}{2! \times (5-2)!}$$

$$= \frac{5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{2} \times 1 \times \cancel{3!}}$$

$$= \frac{10}{1}$$

$$= 10$$

Now this knowledge of combinations can be used to obtain the coefficients instead of Pascal's Triangle.

Expanding $(a+b)^8$ would give:

$$(a+b)^8 = {}^{8}C_0 a^8 b^0 + {}^{8}C_1 a^7 b^1 + {}^{8}C_2 a^6 b^2 + {}^{8}C_3 a^5 b^3 + {}^{8}C_4 a^4 b^4 + {}^{8}C_5 a^3 b^5 + {}^{8}C_6 a^2 b^6 + {}^{8}C_7 a^1 b^7 + {}^{8}C_8 a^0 b^8$$

$$= \frac{8!}{0!\,8!}a^8 + \frac{8!}{1!\,7!}a^7 b + \frac{8!}{2!\,6!}a^6 b^2 + \frac{8!}{3!\,5!}a^5 b^3 + \frac{8!}{4!\,4!}a^4 b^4 + \frac{8!}{5!\,3!}a^3 b^5 + \frac{8!}{6!\,2!}a^2 b^6 + \frac{8!}{7!\,1!}ab^7 + \frac{8!}{8!\,0!}b^8$$

$$= a^8 + 8a^7 b + 28a^6 b^2 + 56a^5 b^3 + 70a^4 b^4 + 56a^3 b^5 + 28a^2 b^6 + 8ab^7 + b^8$$

Expanding $(5-2x)^4$ would give:

$$(5-2x)^4 = {}^{4}C_0 (5)^4 (-2x)^0 + {}^{4}C_1 (5)^3 (-2x)^1 + {}^{4}C_2 (5)^2 (-2x)^2 + {}^{4}C_3 (5)^1 (-2x)^3 + {}^{4}C_4 (5)^0 (-2x)^4$$

$$= \frac{4!}{0!\,4!}625 + \frac{4!}{1!\,3!}125 \times -2x + \frac{4!}{2!\,2!}25 \times (-2)^2 x^2 + \frac{4!}{3!\,1!}5 \times (-2)^3 x^3 + \frac{4!}{4!\,0!}(-2)^4 x^4$$

$$= 625 - 1000x + 600x^2 - 240x^3 + 16x^4$$

In summary:

$$(a+b)^n = {}^{n}C_0 (a)^n (b)^0 + {}^{n}C_1 (a)^{n-1} (b)^1 + {}^{n}C_2 (a)^{n-2} (b)^2 + ... + {}^{n}C_{n-1} (a)^1 (b)^{n-1} + {}^{n}C_n (a)^0 (b)^n$$

Any term can be expressed generally as: $T_{r+1} = {}^{n}C_r (a)^{n-r} (b)^r$

For example: The third term in $(a+b)^9$ is $T_3 = {}^{12}C_2 (a)^{12-2} (b)^2 = {}^{12}C_2 (a)^{10} (b)^2$

- The coefficients ${}^{n}C_r$ are the values from the relevant row of Pascal's Triangle
- The power of a decreases by one and the power of b increases from left to right
- The sum of power a + power b is equal to n for each term.

Determine the 3rd term of $(2-3x)^9$

In the 3rd term of $(2-3x)^9$, $n = 9$ and $r = 2$, so the 3rd term is;

$${}^{9}C_2 (2)^{9-2} (-3x)^2 = \frac{9!}{2!\,7!}128 \times 9x^2 = 41472x^2$$

Calculate the coefficient of x^4 in the expansion of $(3x-1)^9$

By considering the powers of x in the expansion of $(3x-1)^9$, the term containing x^4 can be determined using the general term. From the general term we know $n - r = 4,$ so $9 - r = 4,$ giving $r = 5$ – it is the sixth term.

It is also possible to perform this intuitively, as the powers of x will go $x^9, x^8, x^7, x^6, x^5, x^4 ......$ x^4 is the sixth term.

The sixth term is; $T_6 = {}^9C_5 (3x)^{9-5} (-1)^5 = \dfrac{9!}{5!\,4!} 3^4 \times (-1)^5 x^4 = 126 \times 81 \times -1 x^4 = -10206 x^4$

Expand $\left(x^2 + \dfrac{1}{x} \right)^5$

Because the exponent here is only 5, the Pascal Triangle coefficients are either known or easily determined.

$$\left(x^2 + \dfrac{1}{x} \right)^5 = 1\left(x^2\right)^5 \left(\dfrac{1}{x}\right)^0 + 5\left(x^2\right)^4 \left(\dfrac{1}{x}\right)^1 + 10\left(x^2\right)^3 \left(\dfrac{1}{x}\right)^2 + 10\left(x^2\right)^2 \left(\dfrac{1}{x}\right)^3 + 5\left(x^2\right)^1 \left(\dfrac{1}{x}\right)^4 + 1\left(x^2\right)^0 \left(\dfrac{1}{x}\right)^5$$

$$= x^{10} + 5x^8 \left(\dfrac{1}{x}\right) + 10x^6 \left(\dfrac{1}{x^2}\right) + 10x^4 \left(\dfrac{1}{x^3}\right) + 5x^2 \left(\dfrac{1}{x^4}\right) + \left(\dfrac{1}{x^5}\right)$$

$$= x^{10} + 5x^7 + 10x^4 + 10x + \dfrac{5}{x^2} + \dfrac{1}{x^5}$$

The video below covers some examples similar to those above. Click on the image below to start downloading the video. Depending on your internet connection, this may take a minute or two.

[Video 'Binomial Theorem'](#)

Activity

Expand using Pascal's Triangle

(a)   Expand $(x+3)^4$

(b)   Expand $(2x-3)^6$

(c)   Expand $\left(x+\dfrac{1}{x^2}\right)^4$

(d)   Expand $(2a-b^2)^6$

(e)   Find the 4$^{\text{th}}$ term in $(x-4)^8$

(f)   Find the 5$^{\text{th}}$ term in $\left(\dfrac{2x}{3}+\dfrac{3}{x}\right)^{11}$

(g)   Determine the Coefficient of x^5 in $(2x-1)^{10}$

(h)   Determine the Coefficient of x^3 in $(7-3x)^7$

(i)   Determine the Coefficient of x^2 in $\left(x^2+\dfrac{1}{x}\right)^{10}$

The answers to the activity questions are found in the pdf named:

'Answers to Activity Questions (PCA)'

Answers to activity questions

Multiplying and dividing

1. Expand the following, collect like terms if possible.

(a) $3(9-2x)$

$= 27 - 6x$

(b) $\frac{1}{2}(4x^2 - x - 9)$

$= 2x^2 - \frac{1}{2}x - \frac{9}{2}$

$or \quad 2x^2 - 0.5x - 4.5$

(c) $4x - 3(x-4)$

$= 4x - 3x + 12$

$= x + 12$

(d) $2(x^2 + x + 4) + 3(x+6) - (x^2 - 2x + 9)$

$= 2x^2 + 2x + 8 + 3x + 18 - x^2 + 2x - 9$

$= x^2 + 7x + 17$

(e) $75 - 4(2x^2 + x - 6)$

$= 75 - 8x^2 - 4x + 24$

$= -8x^2 - 4x + 99$

(f) $4(x+4y) - 5(x - 2y + 1) + 2(x-3)$

$= 4x + 16y - 5x + 10y - 5 + 2x - 6$

$= x + 26y - 11$

2. Expand the following, collect like terms if possible.

(a) $a(3a - 4b + 1)$

$= 3a^2 - 4ab + a$

(b) $4ac(b + 2c - 7)$

$= 4abc + 8ac^2 - 28ac$

(c) $p^2\left(2p + 4 + \frac{5}{p}\right)$

$= 2p^3 + 4p^2 + 5p$

(d) $x(x^2 + 4x - 1) - 3(x^2 + 1)$

$= x^3 + 4x^2 - x - 3x^2 - 3$

$= x^3 + x^2 - x - 3$

(e) $-4x(9-x) - \frac{1}{2}(4-x)$

$= -36x + 4x^2 - 2 + \frac{1}{2}x$

$= 4x^2 - 35\frac{1}{2}x - 2$

(f) $\frac{4}{3x}(6x^3 + 5x^2 - x + 4)$

$= 8x^2 + \frac{20}{3}x - \frac{4}{3} + \frac{16}{3x}$

3. Expand the following, collect like terms if possible.

(a) $(2a-1)(4b-3)$

$= 2a(4b-3) - 1(4b-3)$

$= 8ab - 6a - 4b + 3$

(b) $(3x-2)(y+4)$

$= 3xy - 2y + 12x - 8$

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(c) $(x+3)(x-2)$

$= x^2 + x - 6$

(d) $(x-4)(2x+1)$

$= 2x^2 - 7x - 4$

(e) $(x-4)^2$

$= x^2 - 8x + 16$

(f) $(2x-3)(7-x)$

$= 14x - 2x^2 - 21 + 3x$

$= -2x^2 + 17x - 21$

(g) $(x-2)(x+2)$

$= x^2 - 2x + 2x - 4$

$= x^2 - 4$

(h) $(2x+1)(x-3)(x+2)$

$= (2x+1)(x^2 - x - 6)$

$= 2x^3 - 2x^2 - 12x + x^2 - x - 6$

$= 2x^3 - x^2 - 13x - 6$

(i) $(x-2)^3$

$= (x-2)(x-2)(x-2)$

$= (x-2)(x^2 - 4x + 4)$

$= x^3 - 4x^2 + 4x - 2x^2 + 8x - 8$

$= x^3 - 6x^2 + 12x - 8$

(j) $4(2x-1)(x^2 + 7x + 5)$

$= 4(2x^3 + 14x^2 + 10x - x^2 - 7x - 5)$

$= 4(2x^3 + 13x^2 + 3x - 5)$

$= 8x^3 + 52x^2 + 12x - 20$

4. Perform the following divisions.

(a) $(x^2 - 4x + 11) \div (x + 4)$

$$\begin{array}{r} x - 8 + \dfrac{43}{x+4} \\ x+4\overline{\smash{)}\,x^2 - 4x + 11} \\ -\ \underline{x^2 + 4x} \quad \downarrow \\ -8x + 11 \\ -\ \underline{-8x - 32} \\ 43 \end{array}$$

(b) $(x^3 + 4x^2 + x - 6) \div (x - 1)$

$$\begin{array}{r} x^2 + 5x + 6 \\ x-1\overline{\smash{)}\,x^3 + 4x^2 + x - 6} \\ -\ \underline{x^3 - x^2} \quad \downarrow \quad \downarrow \\ 5x^2 + x \quad \downarrow \\ -\ \underline{5x^2 - 5x} \\ 6x - 6 \\ -\ \underline{6x - 6} \\ 0 \end{array}$$

$(x^3 + 4x^2 + x - 6)$

$= (x-1)(x^2 + 5x + 6)$

$= (x-1)(x+2)(x+3)$

(c) $\left(2x^3 - 5x^2 - 14x + 8\right) \div \left(2x - 1\right)$

$$\begin{array}{r} x^2 - 2x - 8 \\ 2x-1\overline{)2x^3 - 5x^2 - 14x + 8} \end{array}$$

$$- \quad \underline{2x^3 - x^2} \qquad \downarrow \qquad \downarrow$$

$$-4x^2 - 14x$$

$$- \quad \underline{-4x^2 + 2x}$$

$$-16x + 8$$

$$- \quad \underline{-16x + 8}$$

$$0$$

$$2x^3 - 5x^2 - 14x + 8$$
$$= \left(2x - 1\right)\left(x^2 - 2x - 8\right)$$
$$= \left(2x - 1\right)\left(x - 4\right)\left(x + 2\right)$$

Factorisation

1. Factorise the following.

(a) $8a + 12 = 4\left(2a + 3\right)$ (b) $6x - 12 = 6\left(x - 2\right)$

(c) $12p - 36 = 12\left(p - 3\right)$ (d) $7a - 21b = 7\left(a - 3b\right)$

(e) $bg + 2bf = b\left(g + 2f\right)$ (f) $14a + 24b = 2\left(7a + 12b\right)$

(g) $a^2 + 2a = a\left(a + 2\right)$ (h) $6x - 6x^2 = 6x\left(1 - x\right)$

(i) $12p^2 - 6p = 6p\left(2p - 1\right)$ (j) $14a - 21a^2 = 7a\left(2 - 3a\right)$

(k) $3g^2 + 2g = g\left(3g + 2\right)$ (l) $12a - 20a^2 = 4a\left(3 - 5a\right)$

(m) $8a^2 + 12ab = 4a\left(2a + 3b\right)$ (n) $9ax^2 - 12a^2x = 3ax\left(3x - 4a\right)$

(o) $a^4b^3 - a^3b^3 = a^3b^3\left(a - 1\right)$ (p) $7a^2b - 49ab^2 = 7ab\left(a - 7b\right)$

(q) $^-2g - 6g^2 = -2g\left(1 + 3g\right)$ (r) $^-4x^3 - 24x = -4x\left(x^2 + 6\right)$

2. Factorise the following.(Grouping factors)

(a) $a\left(x + y\right) + 2\left(x + y\right)$
$$= \left(x + y\right)\left(a + 2\right)$$

(b) $3\left(x + y\right)^2 + 2\left(x + y\right)$
$$= \left(x + y\right)\left[3\left(x + y\right) + 2\right]$$
$$= \left(x + y\right)\left(3x + 3y + 2\right)$$

(c) $a\left(x - y\right) - x + y$
$$= a\left(x - y\right) - \left(x - y\right)$$
$$= \left(x - y\right)\left(a - 1\right)$$

(d) $3a\left(x - y\right) - 2\left(y - x\right)$
$$= 3a\left(x - y\right) + 2\left(x - y\right)$$
$$= \left(x - y\right)\left(3a + 2\right)$$

(e) $4a^4(y-3)-6a^3(y-3)$
$= 2a^3(y-3)(2a-3)$

(f) $3(p+q)^2-6(p+q)$
$= 3(p+q)[(p+q)-2]$
$= 3(p+q)(p+q-2)$

(g) $3a+ab+3c+bc$
$= a(3+b)+c(3+b)$
$= (3+b)(a+c)$

(h) $a^2+3a-ac-3c$
$= a(a+3)-c(a+3)$
$= (a+3)(a-c)$

(i) $a^3-3a^2+ab-3b$
$= a^2(a-3)+b(a-3)$
$= (a-3)(a^2+b)$

(j) $x^2y^2+1-x^2y-y$
$= x^2y^2-x^2y-y+1$
$= x^2y(y-1)-1(y-1)$
$= (y-1)(x^2y-1)$

(k) $2xy^2+2+y^2+4x$
$= 2xy^2+y^2+4x+2$
$= y^2(2x+1)+2(2x+1)$
$= (2x+1)(y^2+2)$

(l) $12x^3+x^2y-4x^3y-3x^2$
$= 12x^3-4x^3y-3x^2+x^2y$
$= 4x^3(3-y)-x^2(3-y)$
$= (3-y)(4x^3-x^2)$
$= x^2(3-y)(4x-1)$

(m) $ax+ay+bx+by$
$= a(x+y)+b(x+y)$
$= (x+y)(a+b)$

(n) $4x^2+x-12ax-3a$
$= x(4x+1)-3a(4x+1)$
$= (4x+1)(x-3a)$

(o) $6x^3-9x^2-2x+3$
$= 3x^2(2x-3)-(2x-3)$
$= (2x-3)(3x^2-1)$

(p) $2a^2b-2a-b+ab^2$
$= 2a^2b-2a+ab^2-b$
$= 2a(ab-1)+b(ab-1)$
$= (ab-1)(2a+b)$

3. Factorise the following.(Common factor - exponent)

(a) $4m^4-2m^2=2m^2(2m^2-1)$

(b) $x^{n+2}+x^{n-2}=x^{n-2}(x^4+1)$

(c) $e^{x+1}+e^x=e^x(e+1)$

(d) $e^{2x}-e^{-2x}=e^{-2x}(e^{4x}-1)$

(e) $e^{4m}+e^{2m}=e^{2m}(e^{2m}+1)$

4. The following questions are partially factorised, complete the missing factor.

(a) $x^2+3x+2=(x+2)(x+1)$

(b) $x^2+5x+6=(x+2)(x+3)$

(c) $x^2+7x+12=(x+3)(x+4)$

(d) $x^2-3x+2=(x-2)(x-1)$

(e) $x^2-7x+10=(x-5)(x-2)$

(f) $x^2-2x-8=(x+2)(x-4)$

(g) $x^2+2x-35=(x+7)(x-5)$

(h) $x^2+4x-12=(x-2)(x+6)$

(i) $x^2-2x-15=(x+3)(x-5)$

(j) $x^2-x-20=(x+4)(x-5)$

(k) $x^2-10x+9=(x-1)(x-9)$

(l) $x^2+15x+56=(x+7)(x+8)$

5. Factorise these trinomials.

(a) $a^2 + 7a + 6 = (a+6)(a+1)$

(b) $p^2 + 9p + 20 = (p+4)(p+5)$

(c) $a^2 - 10a + 24 = (a-6)(a-4)$

(d) $a^2 + 11a + 24 = (a+8)(a+3)$

(e) $x^2 - 15x + 36 = (x-12)(x-3)$

(f) $x^2 + 13x + 36 = (x+4)(x+9)$

(g) $x^2 + 16x - 36 = (x+18)(x-2)$

(h) $2x^2 + 5x - 3 = (2x-1)(x+3)$

(i) $3x^2 - 11x + 6 = (3x-2)(x-3)$

(j) $4x^2 + 25x - 21; ac = -84, b = 25;$
$$28 + (-3) = 25, \quad 28 \times (-3) = -84$$
$$4x^2 + 28x - 3x - 21 = 4x(x+7) - 3(x+7)$$
$$= (x+7)(4x-3)$$

(k) $6x^2 - x - 1 = (3x+1)(2x-1)$

(l) $4a^2 - 12a + 5; ac = 20, b = -12$
$$-10 + (-2) = -12; \quad -10 \times -2 = 20$$
$$4a^2 - 10a - 2a + 5 = 2a(2a-5) - (2a-5)$$
$$= (2a-5)(2a-1)$$

(m) $5y^2 + 13y - 6; ac = -30, b = 13$
$$15 + (-2) = 13; \quad 15 \times (-2) = -30$$
$$5y^2 + 15y - 2y - 6 = 5y(y-3) - 2(y-3)$$
$$= (y-3)(5y-2)$$

(n) $3b^2 + 14b - 5; ac = -15, b = 14$
$$15 + (-1) = 14; \quad 15 \times (-1) = -15$$
$$3b^2 + 15b - b - 5 = 3b(b+5) - (b+5)$$
$$= (b+5)(3b-1)$$

(o) $10x^2 + 9x - 9; ac = -90, b = 9$
$$15 + (-6) = 9; \quad 15 \times (-6) = -90$$
$$10x^2 + 15x - 6x - 9 = 5x(2x+3) - 3(2x+3)$$
$$= (2x+3)(5x-3)$$

(p) $30m^2 - 37m + 10; ac = 300, b = -37$
$$-25 + (-12) = -37; \quad -25 \times -12 = 300$$
$$30m^2 - 25m - 12m + 10 = 5m(6m-5) - 2(6m-5)$$
$$= (6m-5)(5m-2)$$

(q) $6a^2 - 47a + 77; ac = 462, b = -47$
$$-14 + (-33) = -47, \quad -14 \times (-33) = 462$$
$$6a^2 - 14a - 33a + 77 = 2a(3a-7) - 11(3a-7)$$
$$= (3a-7)(2a-11)$$

(r) $8x^2 + 2x - 15; ac = -120, b = 2$
$$12 + (-10) = 2; \quad 12 \times (-10) = -120$$
$$8x^2 + 12x - 10x - 15 = 4x(2x+3) - 5(2x-3)$$
$$= (2x+3)(4x-5)$$

6. Factorise the following by using the difference of squares method.

(a) $b^2 - 49 = (b-7)(b+7)$

(b) $81 - x^2 = (9-x)(9+x)$

(c) $4b^2 - 49 = (2b-7)(2b+7)$

(d) $9x^2 - 4y^2 = (3x-2y)(3x+2y)$

(e) $4a^2 - 25b^2 = (2a-5b)(2a+5b)$

(f) $(x+1)^2 - 49 = \left[(x+1)-7\right]\left[(x+1)+7\right]$
$$= (x-6)(x+8)$$

7. The following questions can be factorised by the perfect square method. Factorise the following by using the perfect square method or alternatively, factorise as trinomials.

(a) $a^2 + 14a + 49 = (a+7)^2$

(b) $x^2 - 12x + 36 = (x-6)^2$

(c) $b^2 - 22b + 121 = (b-11)^2$

(d) $4x^2 + 4x + 1 = (2x+1)^2$

(e) $25a^2 - 20a + 4 = (5a-2)^2$

(f) $4x^2 + 28x + 49 = (2x+7)^2$

8. Factorise the following by using the sum or difference of cubes method.

(a) $b^3 - 8 = (b-2)(b^2 + 2b + 4)$

(b) $64 - x^3 = (4-x)(16 + 4x + x^2)$

(c) $8b^3 - 125 = (2b-5)(4b^2 + 10b + 25)$

(d) $x^3 - 8y^3 = (x-2y)(x^2 + 2xy + 4y^2)$

(e) $27x^3 - 125b^3 = (3x-5b)(9x^2 + 15bx + 25b^2)$

(f) $64x^3 - 1000$
$= (4x-10)(16x^2 + 40x + 100)$

9. Factorise the questions below, one or more of the skills above is required.

(a) $6x^2 - 33x + 36$
$= 3(2x^2 - 11x + 12)$
$= 3(2x-3)(x-4)$

(b) $48x^2 - 3$
$= 3(16x^2 - 1)$
$= 3(4x-1)(4x+1)$

(c) $x^4 + 2x^2 - 3$
$= (x^2 + 3)(x^2 - 1)$
$= (x^2 + 3)(x-1)(x+1)$

(d) $(x-y)^2 - (x+y)^2$
$= [(x-y) + (x+y)][(x-y) - (x+y)]$
$= (2x)(-2y)$
$= -4xy$

(e) $x^2 + 6x + 9 - y^2$
$= (x+3)^2 - y^2$
$= [(x+3) - y][(x+3) + y]$
$= (x-y+3)(x+y+3)$

(f) $x^3 + 3x^2 - xb^2 - 3b^2$
$= x^2(x+3) - b^2(x+3)$
$= (x+3)(x^2 - b^2)$
$= (x+3)(x-b)(x+b)$

(g) $(x+y)^2 + 2(x+y) + 1$
$= A^2 + 2A + 1$ where $A = x+y$
$= (A+1)^2$
$= (x+y+1)^2$

(h) $a^2 - 9 - 4ac + 12c$
$= (a+3)(a-3) - 4c(a+3)$
$= (a+3)[(a-3) - 4c]$
$= (a+3)(a-4c-3)$

(i) $5m^3 - m^2 - 20m + 4$
$= m^2(5m-1) - 4(5m-1)$
$= (5m-1)(m^2 - 4)$
$= (5m-1)(m-2)(m+2)$

(j) $x^6 - 1$
$= (x^3 - 1)(x^3 + 1)$
$= (x-1)(x^2 + x + 1)(x+1)(x^2 - x + 1)$

(k) $4x^2y^2 - x^2 - 4xy - 4y^2$
$= 4x^2y^2 - (x+2y)^2$
$= (2xy)^2 - (x+2y)^2$
$= [2xy - (x+2y)][2xy + (x+2y)]$
$= (2xy - x - 2y)(2xy + x + 2y)$

(l) $2(2a+1)^2 - 8(a-2)^2$
$= 2[(2a+1)^2 - 2^2(a-2)^2]$
$= 2[(2a+1) - 2(a-2)][(2a+1) + 2(a-2)]$
$= 2(5)(4a-1) = 10(4a-1)$

Algebraic fractions

1. Simplify the following

(a) $\dfrac{6a^2bc}{3ab^2} = \dfrac{2ac}{b}$

(b) $\dfrac{12ab}{a^2c} = \dfrac{12b}{ac}$

(c) $\dfrac{10-2t}{5t}$ will not simplify

(d) $\dfrac{a^2b - ab^2}{a-b} = \dfrac{ab\,\cancel{(a-b)}}{\cancel{a-b}} = ab$

(e) $\dfrac{3x^2y - 12y}{x-2} = \dfrac{3y(x^2-4)}{x-2}$

$= \dfrac{3y\,\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= 3y(x+2)$

(f) $\dfrac{4x^2yz - 6xy^2z + 2xyz^2}{2xy} = \dfrac{\cancel{2xy}\,z(2x-3y+z)}{\cancel{2xy}}$

$= z(2x-3y+z)$

(g) $\dfrac{9-3x}{3x-9} = \dfrac{-3(x-3)}{3(x-3)}$

$= -1$

(h) $\dfrac{x^2+2x-15}{3-x} = \dfrac{(x+5)\,\cancel{(x-3)}}{-\cancel{(x-3)}}$

$= -(x+5)$

(i) $\dfrac{x^2+2x-15}{2x^2+7x-15} = \dfrac{(x-3)\,\cancel{(x+5)}}{(2x-3)\,\cancel{(x+5)}}$

$= \dfrac{x-3}{2x-3}$

(j) $\dfrac{4t^2-36}{t^2-3t} = \dfrac{4\,\cancel{(t-3)}(t+3)}{t\,\cancel{(t-3)}}$

$= \dfrac{4(t+3)}{t}$

(k) $\dfrac{x^2+10x+25}{x^2-25} = \dfrac{(x+5)^{\cancel{2}}}{(x-5)\,\cancel{(x+5)}}$

$= \dfrac{x+5}{x-5}$

(l) $\dfrac{6+7x-3x^2}{3x^2-x-2} = \dfrac{-\cancel{(3x+2)}(x-3)}{\cancel{(3x+2)}(x-1)}$

$= \dfrac{-(x-3)}{x-1}$

$= \dfrac{3-x}{x-1}$

2. Perform the addition or subtraction. Simplify as much as possible.

(a) $\dfrac{4}{m} + \dfrac{2}{n} = \dfrac{4n+2m}{mn}$

(b) $\dfrac{n}{m} + \dfrac{2}{3m} = \dfrac{3n+2}{3m}$

(c) $\dfrac{2}{x} - \dfrac{1}{4x} - \dfrac{3}{5x} = \dfrac{40-5-12}{20x} = \dfrac{27}{20x}$

(d) $\dfrac{1}{x^2} + \dfrac{2}{4x} + \dfrac{1}{5} = \dfrac{10+5x+2x^2}{10x^2}$

(e) $\dfrac{3}{x+2} + \dfrac{x}{x-2} = \dfrac{3(x-2)+x(x+2)}{(x+2)(x-2)}$

$= \dfrac{x^2+5x-6}{(x+2)(x-2)}$

$= \dfrac{(x+6)(x-1)}{(x+2)(x-2)}$

(f) $\dfrac{x-2}{4} - \dfrac{x+1}{3} = \dfrac{3(x-2)-4(x+1)}{12}$

$= \dfrac{-x-10}{12}$

(g)
$$\frac{2(x+2)}{5}+\frac{3(4-x)}{4}=\frac{8(x+2)+15(4-x)}{20}$$
$$=\frac{-7x+76}{20}$$

(h)
$$y^2-\frac{y+2}{y}=\frac{y^3-(y+2)}{y}$$
$$=\frac{y^3-y-2}{y}$$

(i)
$$\frac{2-x}{x^2-16}+\frac{3}{x+4}=\frac{(2-x)+3(x-4)}{(x-4)(x+4)}$$
$$=\frac{2x-10}{(x-4)(x+4)}$$

(j)
$$\frac{x-3}{x^2-4x-5}-\frac{2(2x+1)}{2x^2-11x+5}$$
$$=\frac{x-3}{(x-5)(x+1)}-\frac{4x+2}{(2x-1)(x-5)}$$
$$=\frac{(x-3)(2x-1)-(x+1)(4x+2)}{(x-5)(x+1)(2x-1)}$$
$$=\frac{-2x^2-13x+1}{(x-5)(x+1)(2x-1)}$$

(k)
$$\frac{1}{x^3-9x}+\frac{4}{x-3}-\frac{5}{x+3}$$
$$=\frac{1+4x(x+3)-5x(x-3)}{x(x-3)(x+3)}$$
$$=\frac{-x^2+27x+1}{x(x-3)(x+3)}$$

(l)
$$\frac{x^2-5x}{2x^2+7x+3}+\frac{4x^2-1}{x^2-2x-15}$$
$$=\frac{x(x-5)}{(2x+1)(x+3)}+\frac{4x^2-1}{(x-5)(x+3)}$$
$$=\frac{x(x-5)(x-5)+(4x^2-1)(2x+1)}{(2x+1)(x-5)(x+3)}$$
$$=\frac{(x^3-10x^2+25x)+(8x^3+4x^2-2x-1)}{(2x+1)(x-5)(x+3)}$$
$$=\frac{9x^3-6x^2+23x-1}{(2x+1)(x-5)(x+3)}$$

3. Simplify the following

(a)
$$\frac{7a^2b}{5c}\times\frac{2ac}{b^2}=\frac{7a^3bc}{5b^2c}$$
$$=\frac{7a^3}{5b}$$

(b)
$$\frac{4a^2b}{3c}\div\frac{2a}{3b}=\frac{4a^2b}{3c}\times\frac{3b}{2a}$$
$$=\frac{12a^2b^2}{6ac}$$
$$=\frac{2ab^2}{c}$$

(c)
$$\frac{3x+9}{5}\times\frac{10}{x+3}=\frac{30\cancel{(x+3)}}{5\cancel{(x+3)}}$$
$$=6$$

(d)
$$\frac{16x^2y^2z^4}{5x}\div\frac{x^2}{2}=\frac{16x^2y^2z^4}{5x}\times\frac{2}{x^2}$$
$$=\frac{32x^2y^2z^4}{5x^3}$$
$$=\frac{32y^2z^4}{5x}$$

(e) $\dfrac{x^2-4x}{x^2-4}\times\dfrac{x^2+4x+4}{x^2-2x-8}$

$=\dfrac{x\,(x-4)\,(x+2)^2}{(x-2)\,(x+2)\,(x-4)\,(x+2)}$

$=\dfrac{x}{x-2}$

(f) $\dfrac{x^2+3x+2}{x^2-7x+12}\div\dfrac{2x^2+8x+8}{x^2+7x+6}$

$=\dfrac{x^2+3x+2}{x^2-7x+12}\times\dfrac{x^2+7x+6}{2x^2+8x+8}$

$=\dfrac{(x+1)(x+2)}{(x-3)(x-4)}\times\dfrac{(x+6)(x+1)}{2(x+2)^2}$

$=\dfrac{(x+6)(x+1)^2}{2(x+2)(x-3)(x-4)}$

4. Simplify the following

(a) $\dfrac{2-\dfrac{1}{x}}{\dfrac{2}{x}-5}$

$=\dfrac{\dfrac{2x-1}{x}}{\dfrac{2-5x}{x}}$

$=\dfrac{2x-1}{\not x}\times\dfrac{\not x}{2-5x}$

$=\dfrac{2x-1}{2-5x}$

(b) $\dfrac{\dfrac{3}{x-2}+1}{5-\dfrac{1}{x+2}}$

$=\dfrac{\dfrac{3+(x-2)}{x-2}}{\dfrac{5(x+2)-1}{x+2}}$

$=\dfrac{x+1}{x-2}\times\dfrac{x+2}{5x+9}$

$=\dfrac{(x+1)(x+2)}{(x-2)(5x+9)}$

Binomial expansions

1. Expand using Pascal's Triangle

(a) Expand $(x+3)^4$

$(x+3)^4=1x^4+4x^3(3)+6x^2(3)^2+4x(3)^3+1(3)^4$
$=x^4+12x^3+54x^2+108x+81$

(b) Expand $(2x-3)^6$

$(2x-3)^6=1(2x)^6+6(2x)^5(-3)+15(2x)^4(-3)^2+20(2x)^3(-3)^3+15(2x)^2(-3)^4+6(2x)(-3)^5+1(-3)^6$
$=64x^6-576x^5+2160x^4-4320x^3+4869x^2-2916x+729$

(c) Expand $\left(x+\dfrac{1}{x^2}\right)^4$

$\left(x+\dfrac{1}{x^2}\right)^4=1x^4+4x^3\left(\dfrac{1}{x^2}\right)+6x^2\left(\dfrac{1}{x^2}\right)^2+4x\left(\dfrac{1}{x^2}\right)^3+1\left(\dfrac{1}{x^2}\right)^4$

$=x^4+4x+\dfrac{6}{x^2}+\dfrac{4}{x^5}+\dfrac{1}{x^8}$

(d) Expand $(2a-b^2)^6$

$$(2a-b^2)^6 = 1(2a)^6 + 6(2a)^5(-b^2) + 15(2a)^4(-b^2)^2 + 20(2a)^3(-b^2)^3 + 15(2a)^2(-b^2)^4 + 6(2a)(-b^2)^5 + 1(-b^2)^6$$
$$= 64a^6 - 192a^5b^2 + 240a^4b^4 - 160a^3b^6 + 60a^2b^8 - 12ab^{10} + b^{12}$$

(e) Find the 4$^{\text{th}}$ term in $(x-4)^8$

$$T_{r+1} = {}^nC_r(a)^{n-r}(b)^r$$
$$T_4 = {}^8C_3(x)^{8-3}(-4)^3 = \frac{8!}{3!5!}x^5(-64) = -\frac{8\times7\times6}{6}\times64x^5 = -3584x^5$$

(f) Find the 5$^{\text{th}}$ term in $\left(\frac{2x}{3}+\frac{3}{x}\right)^{11}$

$$T_{r+1} = {}^nC_r(a)^{n-r}(b)^r$$
$$T_5 = {}^{11}C_4\left(\frac{2x}{3}\right)^{11-4}\left(\frac{3}{x}\right)^4$$
$$= \frac{11!}{4!7!}\left(\frac{2x}{3}\right)^7\left(\frac{3}{x}\right)^4$$
$$= 330\times\frac{128x^7}{3^7}\times\frac{3^4}{x^4} = \frac{14080}{9}x^3$$

(g) Determine the Coefficient of x^5 in $(2x-1)^{10}$

The x^5 term is the 6$^{\text{th}}$ term (*n – r = 5, as n = 10, r=5, it is the 6th term*)
$$T_{r+1} = {}^nC_r(a)^{n-r}(b)^r$$
$$T_6 = {}^{10}C_5(2x)^{10-5}(-1)^5$$
$$= -\frac{10!}{5!5!}\times32x^5$$
$$= -252\times32x^5$$
$$= -8064x^5$$

(h) Determine the Coefficient of x^3 in $(7-3x)^7$

The x^3 term is the 4$^{\text{th}}$ term (*as r=3, it is the 4th term*)
$$T_{r+1} = {}^nC_r(a)^{n-r}(b)^r$$
$$T_4 = {}^7C_3(7)^{7-3}(-3x)^3$$
$$= \frac{7!}{3!4!}(2401)(-27x^3)$$
$$= -35\times2401\times27x^3 = -2268945x^3$$

(i) Determine the Coefficient of x^2 in $\left(x^2+\frac{1}{x}\right)^{10}$

The x^2 term is the 7$^{\text{th}}$ term (*2(n – r)-r = 2, as n = 10, 20-2r-r=2, then r=6*)
$$T_{r+1} = {}^nC_r(a)^{n-r}(b)^r$$
$$T_7 = {}^{10}C_6(x^2)^{10-6}\left(\frac{1}{x}\right)^6$$
$$= \frac{10!}{6!4!}x^8x^{-6}$$
$$= 210x^2$$